19.5.5.1

19.5.5

1) $$\psi_{\vec{k}}(\vec{r}) = \sum_{l=0}^{\infty} R_{kl}(r)\, P_l(\cos\theta) \quad \text{for all } r$$

$$= e^{ikz} + \frac{f(\theta)}{r} e^{ikr} \quad \text{for } r \gg a$$

$$\rightarrow \sum_{l=0}^{\infty} \frac{1}{r}\left(A_{l+}(k)\, e^{ikr} + A_{l-}(k)\, e^{-ikr} \right) P_l(\cos\theta) \quad \text{as } r \to \infty$$

outgoing incoming

$$S_l(k) = \frac{A_{l+}(k)}{A_{l-}(k)} (-1)^{l+1} = \eta(k)\, e^{2i\delta_l(k)}$$

with $\eta(k) < 1$ because of absorption.

since

$$e^{ikz} = \sum_{l=0}^{\infty} i^l (2l+1)\, j_l(kr)\, P_l(\cos\theta)$$

$$\rightarrow \sum_{l=0}^{\infty} i^l (2l+1) \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right) P_l(\cos\theta)$$

19.5.5.2

$$\sum_{l=0}^{\infty} \left(A_{l+}(k)\, e^{ikr} + A_{l-}(k)\, e^{-ikr} \right) P_l(\cos\theta)$$

$$= f(\theta)\, e^{ikr}$$

$$+ \sum_{l=0}^{\infty} i^l (2l+1) \frac{1}{2ik} \left(e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)} \right) \cdot P_l(\cos\theta)$$

$$\therefore \quad A_{l-}(k) = i^l (2l+1) \left(\frac{-1}{2ik} \right) i^l$$

$$= \frac{(-1)^{l+1}}{2ik} (2l+1)$$

Let $$f(\theta) = \sum_{l=0}^{\infty} f_l(k)\, P_l(\cos\theta)$$

$$A_{l+}(k) = f_l(k) + \cancel{i^l} (2l+1) \frac{1}{2ik} \cancel{(-i)^l}$$

$$f_l(k) = A_{l+}(k) - \frac{1}{2ik}(2l+1)$$

$$= (-1)^{l+1} \left(S_l(k) - 1 \right) A_{l-}(k)$$

$$f_\ell(k) = \frac{1}{2ik}(2\ell+1)\left(S_\ell(k)-1\right)$$

$$f(\theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ik}\left(S_\ell(k)-1\right) P_\ell(\cos\theta)$$

$$\sigma_{\text{elastic}} = \int d\Omega \, |f(\theta)|^2$$

$$= 2\pi \sum_{\ell=0}^{\infty} \frac{(2\ell+1)^2}{4k^2} |S_\ell(k)-1|^2 \frac{2}{2\ell+1}$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(\eta_\ell e^{2i\delta_\ell}-1\right)\left(\eta_\ell e^{-2i\delta_\ell}-1\right)$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1+\eta_\ell^2 - 2\eta_\ell \cos(2\delta_\ell)\right).$$

$$\sigma_{\text{inel}} = \frac{\#\text{ particles absorbed per unit time}}{\text{incident flux}}$$

$$\text{incident flux} = \frac{\hbar k}{\mu} \qquad \text{(as before)}$$

19.5.5.4

$$\vec{j} = \frac{\hbar}{\mu} \, \mathrm{Im} \left(\psi^{*}_{\vec{k}}(\vec{r}) \, \vec{\nabla} \, \psi_{\vec{k}}(\vec{r}) \right)$$

$$\hat{r} \cdot \vec{j} \rightarrow \frac{\hbar k}{\mu \, r^2} \sum_{l, l' = 0}^{\infty} \Big(\underbrace{|A_{l+}(k)|^2}_{\text{outgoing}} - \underbrace{|A_{l-}(k)|^2}_{\text{incoming}} \Big) \cdot P_l(\cos\theta) \, P_{l'}(\cos\theta) + o\left(\frac{1}{r^3}\right)$$

particles absorbed per unit time

$$= - \int d\Omega \; r^2 \; \hat{r} \cdot \vec{j}$$

$$= \frac{\hbar k}{\mu} \sum_{l=0}^{\infty} \left(|A_{l-}(k)|^2 - |A_{l+}(k)|^2 \right) 2\pi \frac{2}{2l+1}$$

$$= j_{inc} \; 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1} \left(1 - \eta_l^2 \right) |A_{l-}(k)|^2$$

$$= j_{inc} \; 4\pi \sum_{l=0}^{\infty} (2l+1) \frac{1}{4k^2} \left(1 - \eta_l^2 \right)$$

$$\therefore \quad \sigma_{inel} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left(1 - \eta_l^2 \right) .$$

19.5.5.5

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel}$$

$$= \frac{2\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)\left(1 - \eta_l \cos(2\delta_l)\right)$$

$$\operatorname{Im} f(0) = \operatorname{Im} \sum_{l=0}^{\infty} \frac{2l+1}{2ik} \left(\eta_l e^{2i\delta_l} - 1\right) \underbrace{P_l(1)}_{=1}$$

$$= \frac{1}{2k} \sum_{l=0}^{\infty} (2l+1)\left(1 - \eta_l \cos(2\delta_l)\right)$$

$$\therefore \quad \sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(0).$$

2)

$$\sigma_{inel} \simeq \frac{\pi}{k^2}\left(\int_0^{kr_0} dl \cdot 2l \cdot 1 + \int_{kr_0}^{\infty} dl \cdot 2l \cdot 0\right)$$

$$\simeq \frac{\pi}{k^2} (kr_0)^2 = \pi r_0^2$$

$$\sigma_{el} \simeq \frac{\pi}{k^2}\left(\int_0^{kr_0} dl \cdot 2l \cdot 1 + \int_{kr_0}^{\infty} dl \cdot 2l \cdot 0\right)$$

19.5.5.6

$$\sigma_{el} \simeq \pi r_0^2 .$$

If $\sigma_{inel} \neq 0$, then $\sigma_{tot} \neq 0$ and hence $\operatorname{Im} f(0) \neq 0$ by the optical theorem. Since $f(0) \neq 0$ there is necessarily scattering in the forward direction.

20.1.1.1

20.1.1

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(c\,\vec{\alpha} \cdot \frac{\hbar}{i} \vec{\nabla} + \beta m c^2 \right) \Psi$$

$$\frac{\partial \Psi}{\partial t} = \left(-c\,\vec{\alpha} \cdot \vec{\nabla} - \frac{i m c^2}{\hbar} \beta \right) \Psi$$

$$\frac{\partial \Psi^\dagger}{\partial t} = \Psi^\dagger \left(-c\,\vec{\alpha}^\dagger \cdot \overleftarrow{\nabla} + \frac{i m c^2}{\hbar} \beta^\dagger \right)$$

$$= \Psi^\dagger \left(-c\,\vec{\alpha} \cdot \overleftarrow{\nabla} + \frac{i m c^2}{\hbar} \beta \right)$$

since $\vec{\alpha}^\dagger = \vec{\alpha}$ and $\beta^\dagger = \beta$.

By $\overleftarrow{\nabla}$ we mean that the gradient operator acts on the left

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \qquad \vec{\nabla} \Psi = \begin{pmatrix} \vec{\nabla} \Psi_1 \\ \vec{\nabla} \Psi_2 \\ \vec{\nabla} \Psi_3 \\ \vec{\nabla} \Psi_4 \end{pmatrix}$$

$$\Psi^\dagger = \left(\Psi_1^* \quad \Psi_2^* \quad \Psi_3^* \quad \Psi_4^* \right)$$

20.1.1.2

$$\psi^{\dagger}\overleftarrow{\nabla} = \left(\vec{\nabla}\psi_1^{*} \quad \vec{\nabla}\psi_2^{*} \quad \vec{\nabla}\psi_3^{*} \quad \vec{\nabla}\psi_4^{*} \right)$$

$$\frac{\partial}{\partial t}\left(\psi^{\dagger}\psi\right) = \psi^{\dagger}\left(-c\,\vec{\alpha}\cdot\vec{\nabla} - \frac{imc^2}{\hbar}\beta\right)\psi$$

$$+ \psi^{\dagger}\left(-c\,\vec{\alpha}\cdot\overleftarrow{\nabla} + \frac{imc^2}{\hbar}\beta\right)\psi$$

$$= -c\left(\psi^{\dagger}\vec{\alpha}\cdot\vec{\nabla}\psi + \psi^{\dagger}\vec{\alpha}\cdot\overleftarrow{\nabla}\psi\right)$$

$$= -c\,\vec{\nabla}\cdot\left(\psi^{\dagger}\vec{\alpha}\psi\right)$$

$\therefore$ $$\frac{\partial P}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$ with

$$P = \psi^{\dagger}\psi$$ and

$$\vec{j} = c\,\psi^{\dagger}\vec{\alpha}\psi \,.$$

20.2.1.1

20.2.1

$$\vec{\pi} \times \vec{\pi} \quad \psi$$

$$= \hat{k}\, \epsilon_{j\ell k} \left(\frac{\hbar}{i} \partial_j - \frac{q}{c} A_j \right) \left(\frac{\hbar}{i} \partial_\ell - \frac{q}{c} A_\ell \right) \psi$$

$$= \hat{k} \left(-\frac{\hbar q}{ic} \right) \epsilon_{j\ell k} \left[\partial_j (A_\ell \psi) - A_j \partial_\ell \psi \right]$$

$$= \hat{k} \left(-\frac{\hbar q}{ic} \right) \epsilon_{j\ell k} \left(\partial_j A_\ell \right) \psi$$

$$= -\frac{\hbar q}{ic} B_k \hat{k} \psi$$

$$\therefore \quad \vec{\pi} \times \vec{\pi} = \frac{i\hbar q}{c} \vec{B}$$

20.2.2.1

20.2.2 $\left[c\vec{\alpha}\cdot\left(\frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}\right) + \beta mc^2 + q\phi \right]\psi$

$= E\psi$

For $\psi = \begin{pmatrix} \chi \\ \Phi \end{pmatrix}$ and $\phi = 0$,

we have [Eqs. (20.2.7) and (20.2.8)]

$(E - \mu c^2)\chi = c\,\vec{\sigma}\cdot\vec{\pi}\,\Phi$

$(E + \mu c^2)\Phi = c\,\vec{\sigma}\cdot\vec{\pi}\,\chi$

with $\vec{\pi} = \frac{\hbar}{i}\vec{\nabla} - \frac{q}{c}\vec{A}$

$\therefore$ $(E^2 - \mu^2 c^4)\chi = c\,\vec{\sigma}\cdot\vec{\pi}\,(E + \mu c^2)\Phi$

$= c^2(\vec{\sigma}\cdot\vec{\pi})^2\chi$

$= c^2\left(\vec{\pi}\cdot\vec{\pi} - \frac{q\hbar}{c}\vec{\sigma}\cdot\vec{B}\right)\chi$

using Eqs. (20.2.15) and (20.2.16)

For $\vec{B} = B_0\hat{z}$

$\vec{A} = \frac{1}{2}B_0(-y\hat{x} + x\hat{y})$

20.2.2.2

$$\left(E^2 - \mu^2 c^4\right) \chi$$

$$= c^2 \left[-\hbar^2 \frac{\partial^2}{\partial z^2} + \left(\frac{\hbar}{i} \frac{\partial}{\partial x} + \frac{qB_0}{2c} y \right)^2 \right.$$

$$\left. + \left(\frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{qB_0}{2c} x \right)^2 - \frac{q\hbar}{c} \vec{\sigma} \cdot \vec{B} \right] \chi$$

$$= c^2 \left[-\hbar^2 \nabla^2 + \frac{qB_0 \hbar}{ic} \left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) \right.$$

$$\left. + \left(\frac{qB_0}{2c} \right)^2 (x^2 + y^2) - \frac{q\hbar B_0}{c} \sigma_z \right] \chi$$

Let

$$\chi = e^{\frac{i}{\hbar} p_z z} \begin{pmatrix} \chi_\uparrow (x,y) \\ \chi_\downarrow (x,y) \end{pmatrix}$$

Then

$$\frac{1}{c^2} \left(E^2 - \mu^2 c^4 - c^2 p_z^2 \right) \chi_{\uparrow\downarrow}$$

$$= \left[-\hbar^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \left(\frac{qB_0}{2c} \right)^2 (x^2 + y^2) \right.$$

$$\left. - \frac{qB_0}{c} \left(L_z \pm \hbar \right) \right] \chi_{\uparrow\downarrow}$$

See the solutions to previous homework problems 12.3.7 and 12.3.8

$$H' = \frac{1}{2\mu}\left(P_x^2 + P_y^2\right) + \frac{1}{2}\mu\omega^2\left(x^2 + y^2\right)$$

is invariant under two dimensional rotations

$$[H', L_z] = 0$$

Its eigenvalues are

$$E' = (2k + |m| + 1)\hbar\omega$$

with $k = 0, 1, 2 \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$

$m\hbar$ is the eigenvalue of L_z.

The eigenvalues of

$$-\hbar^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{qB_0}{2c}\right)^2 (x^2 + y^2) - \frac{qB_0}{c} L_z$$

20.2.2.4

are therefore

$$2\mu\left[(2k+|m|+1)\hbar\omega - \omega\hbar m\right]$$

with $\omega = \dfrac{qB_0}{2\mu c}$.

Note that this consistent with the answer to problem 12.3.8

$$(2k+|m|-m+1)\hbar\omega$$

$$= (n+\tfrac{1}{2})\hbar\omega_0 \quad \text{with}$$

$$\omega_0 = \frac{qB_0}{\mu c} = 2\omega \quad \text{and}$$

$$n = k + \frac{1}{2}(|m|-m)$$

$$= 0, 1, 2, 3 \ldots$$

In conclusion,

20.2.2.5

$$E^2 = \mu^2 c^4 + c^2 p_z^2$$
$$+ 2\mu c^2 \left[2k + (|m| - m) + 1 - 2m_s \right] \hbar\omega$$

$$E = + \left[\mu^2 c^4 + c^2 p_z^2 + 2\mu c^2 \left[2k + (|m| - m) + 1 - 2m_s \right] \hbar\omega \right]^{1/2}$$

where $k = 0, 1, 2 \ldots$

$m = 0, \pm 1, \pm 2, \ldots$

$m_s = \pm \frac{1}{2}$

$\hbar m$ is the eigenvalue of L_z

$\hbar m_s$ " " " " S_z.

21.1.3

$$a|0\rangle = 0$$

$$\langle x, y| a |0\rangle$$

$$= \langle x, y| \left[\sqrt{\frac{\mu\omega_0}{2\hbar}} \left(\frac{c}{qB} P_x + \frac{1}{2} y \right) + i \frac{1}{\sqrt{2\mu\omega_0\hbar}} \left(P_y - \frac{qB}{2c} x \right) \right] |0\rangle$$

$$= \frac{1}{\sqrt{2\mu\omega_0\hbar}} \langle x, y| (P_x + i P_y) - i \frac{qB}{2c} (x + iy) |0\rangle$$

$$= 0$$

$$\left[\frac{\hbar}{i} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) - i \frac{qB}{2c} (x + iy) \right] \psi_0(x,y) = 0$$

$$z = x + iy \qquad \frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \frac{1}{2}$$

$$\frac{\partial}{\partial z^*} = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \frac{1}{2}$$

21.1.3.2

so that $\frac{\partial}{\partial z} z = 1$, $\frac{\partial}{\partial z} z^* = 0$,

$$\left(\frac{\partial}{\partial z^*} + \frac{qB}{4c\hbar} z \right) \psi_0(z, z^*) = 0$$

Let

$$\psi_0(z, z^*) = e^{-\frac{qB}{4c\hbar} z z^*} u(z, z^*)$$

$$\frac{\partial}{\partial z^*} \psi_0(z, z^*) = e^{-\frac{qB}{4c\hbar} z z^*} \cdot \left(\frac{-qB}{2c\hbar} z + \frac{\partial}{\partial z^*} \right) u(z, z^*)$$

$$= -\frac{qB}{2c\hbar} e^{-\frac{qB}{4c\hbar}} u(z, z^*)$$

$$\therefore \quad \frac{\partial}{\partial z^*} u(z, z^*) = 0$$

$$\therefore \quad \psi_0(x, y) = \sum_m c_m (x + iy)^m e^{-\frac{qB}{4\hbar c}(x^2 + y^2)}.$$

arbitrary coefficients

Let $\psi_{0,m} = z^m e^{-\frac{qB}{4\hbar c} z z^*}$

$$|\psi_{0,m}|^2 = (x^2+y^2)^m e^{-\frac{qB}{2\hbar c}(x^2+y^2)}$$

$$= r^{2m} e^{-\frac{qB}{2\hbar c} r^2}$$

For large m, most of the support of $|\psi_{0,m}|^2$ is at r_m:

$$\frac{d}{dr} |\psi_{0,m}|^2 \Big|_{r=r_m} = |\psi_{0,m}|^2 \left(\frac{2m}{r} - \frac{qB}{\hbar c} r \right) \Big|_{r=r_m} = 0$$

$$\therefore \quad r_m = \sqrt{\frac{2m\hbar c}{qB}} = \sqrt{2m}\, r_0$$

with $r_0 = \sqrt{\frac{c\hbar}{qB}}$.

If the available volume is a disc of radius R, the maximum value of m is N such that

21.1.3.4

$$r_N = \sqrt{2N}\, r_0 = R$$

$$\therefore \quad N = \frac{R^2}{2 r_0^2} = \frac{q B R^2}{2 \hbar c}$$

$$= \frac{\pi R^2 B}{\Phi_0} \qquad \text{with}$$

$$\Phi_0 = \frac{2\pi \hbar c}{q} .$$