19.3.1

In the Born approximation, for the Yukawa potential

$$V(r) = \frac{g\, e^{-\mu_0 r}}{r}$$

the differential scattering cross-section is [cfr. Eq. (19.3.11)]

$$\frac{d\sigma_Y}{d\Omega} = \frac{4\mu^2 g^2}{\hbar^4 \left[\mu_0^2 + 4k^2 \sin^2 \theta/2\right]^2}$$

where μ is the mass of the particle being scattered and $\frac{\hbar^2 k^2}{2\mu}$ is its energy (the scatterer does not recoil).

$$\sigma_Y = \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \; \frac{d\sigma_Y}{d\Omega}$$

$$= \frac{4\mu^2 g^2}{\hbar^2} \, 2\pi \int_{-1}^{+1} dx \, \frac{1}{\left[\mu_0^2 + 2k^2(1-x)\right]^2}$$

where we have substituted $x = \cos\theta$.

$$\sigma_y = \frac{4\mu^2 g^2}{\hbar^2} 2\pi \left. \frac{1}{\mu_0^2 + 2k^2(1-x)} \right|_{-1}^{+1} \frac{1}{2k^2}$$

$$= \frac{4\mu^2 g^2}{\hbar^2} 2\pi \frac{1}{2k^2} \left[\frac{1}{\mu_0^2} - \frac{1}{\mu_0^2 + 4k^2} \right]$$

$$= \frac{16\pi \mu^2 g^2}{\hbar^2 \mu_0^2 (\mu_0^2 + 4k^2)}$$

$$= 16\pi r_0^2 \left(\frac{g\mu r_0}{\hbar} \right)^2 \frac{1}{1+4k^2 r_0^2}$$

where $r_0 = \frac{1}{\mu_0}$ is the range.

The geometrical cross section associated with this range is πr_0^2.

The Born approximation's validity requires:

$$\frac{\sigma_y}{\pi r_0^2} = 4 \left(\frac{g\mu r_0}{\hbar} \right)^2 \frac{1}{1+4k^2 r_0^2} \ll 1 .$$

S 19.3.2 $i.$ $V(r) = -V_0\, \Theta(r_0 - r)$

In the Born approximation,

$$f(\theta) = \frac{-\mu}{2\pi \hbar^2} \int e^{-i\vec{q}\cdot\vec{r}}\, V(\vec{r})\, d^3r$$

$$= \frac{\mu V_0}{2\pi \hbar^2} \int_0^{r_0} r^2 dr \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, e^{iqr\cos\theta}$$

$$= \frac{\mu V_0}{\hbar^2} \int_0^{r_0} r^2 dr\, \frac{1}{iqr}\left(e^{iqr} - e^{-iqr}\right)$$

$$= \frac{\mu V_0}{\hbar^2 q}\, 2 \int_0^{r_0} r\, dr\, \sin qr$$

$$= -\frac{\mu V_0}{\hbar^2 q}\, 2\, \frac{d}{dq} \int_0^{r_0} dr\, \cos qr$$

$$= -\frac{2\mu V_0}{\hbar^2 q}\, \frac{d}{dq}\, \frac{1}{q} \sin qr \Big|_0^{r_0}$$

$$= -\frac{2\mu V_0}{\hbar^2 q} \left(\frac{-1}{q^2} \sin qr_0 + \frac{r_0}{q} \cos qr_0 \right)$$

$$f(\theta) = \frac{2\mu V_0}{\hbar^2 q^3}\left(\sin qr_0 - r_0 q \cos qr_0\right)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2\mu V_0}{\hbar^2 q^3}\right)^2 \left(\sin qr_0 - qr_0 \cos qr_0\right)^2$$

$$= 4 r_0^2 \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 \frac{\left(\sin qr_0 - qr_0 \cos qr_0\right)^2}{(qr_0)^6} .$$

ii. $$q = 2k \sin \theta/2$$

As $kr_0 \to 0$, $qr_0 \to 0$ and

$$\frac{\sin qr_0 - qr_0 \cos qr_0}{(qr_0)^3}$$

$$= \frac{1}{(qr_0)^3}\Big\{\cancel{qr_0} - \frac{1}{3!}(qr_0)^3 + O(qr_0)^5$$

$$-(qr_0)\left[\cancel{1} - \frac{1}{2!}(qr_0)^2 + O(qr_0)^4\right]\Big\}$$

$$\to -\frac{1}{3!} + \frac{1}{2!} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

$$\frac{d\sigma}{d\Omega} \to 4 r_0^2 \left(\frac{\mu V_0 r_0^2}{\hbar^2} \right)^2 \frac{1}{9}$$

which is independent of θ.

Thus the scattering is isotropic in the $kr_0 \to 0$ limit and

$$\sigma = 4\pi \frac{d\sigma}{d\Omega} = \frac{16\pi r_0^2}{9} \left(\frac{\mu V_0 r_0^2}{\hbar^2} \right)^2 .$$

19.3.3

For $V(r) = V_0 \, e^{-r^2/r_0^2}$,

in the Born approximation

$$f(\theta) = \frac{-\mu}{2\pi\hbar^2} \int d^3r \; V(\vec{r}) \, e^{-i\vec{q}\cdot\vec{r}}$$

$$(\hbar\vec{q} = \vec{p}_f - \vec{p}_i)$$

$$= \frac{-2\mu}{\hbar^2} \int_0^\infty r\,dr \; \frac{\sin(qr)}{q} \; V_0 \, e^{-r^2/r_0^2}$$

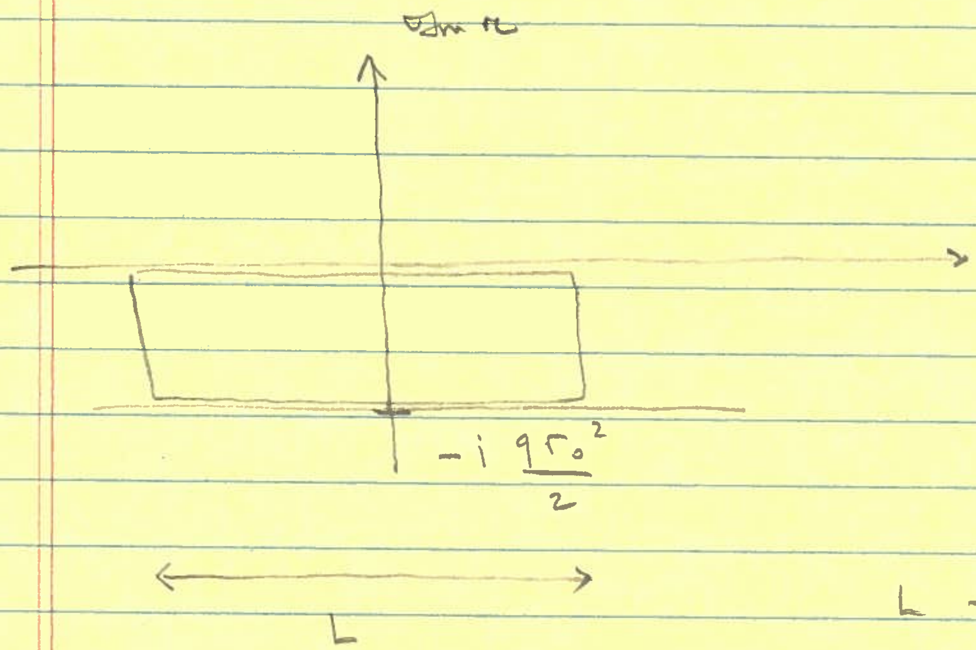
$$= -\frac{2\mu}{\hbar^2 q} V_0 \int_0^\infty r\,dr \; \frac{1}{2i} \left(e^{-r^2/r_0^2 + iqr} - e^{-r^2/r_0^2 - iqr} \right)$$

$$= \frac{i\mu V_0}{\hbar^2 q} \int_{-\infty}^{+\infty} r\,dr \; e^{-r^2/r_0^2 + iqr}$$

$$= \frac{i\mu V_0}{\hbar^2 q} \int_{-\infty}^{+\infty} r\,dr \; e^{-\frac{1}{r_0^2}\left(r - i\frac{q r_0^2}{2}\right)^2 - \frac{q^2 r_0^2}{4}}$$

$$= \frac{i \mu V_0}{\hbar^2 q} e^{-\frac{q^2 r_0^2}{4}} \left[\int_{-\infty}^{+\infty} dr \left(r - i \frac{q r_0^2}{2} \right) e^{-\frac{1}{r_0^2}\left(r - i \frac{q r_0^2}{2}\right)^2} \right.$$

$$\left. + \frac{i q r_0^2}{2} \int_{-\infty}^{+\infty} dr \; e^{-\frac{1}{r_0^2}\left(r - i \frac{q r_0^2}{2}\right)^2} \right]$$



$L \to \infty$

By contour integration, the integral can be shifted from

$$r - i \frac{q r_0^2}{2} \quad \text{to} \quad r$$

$$\int_{-\infty}^{+\infty} dr \; r \, e^{-\frac{1}{r_0^2} r^2} = 0$$ because the integrand is odd

$$\int_{-\infty}^{+\infty} dr \; e^{-1/r_0^2 \, r^2} = r_0\sqrt{\pi}$$

$$\therefore \quad f(\theta) = \frac{-\mu V_0}{2\hbar^2} r_0^3 \sqrt{\pi} \; e^{-\frac{1}{4} q^2 r_0^2} .$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{\pi \mu^2 V_0^2 r_0^6}{4\hbar^4} \; e^{-\frac{1}{2} q^2 r_0^2}$$

$$\sigma = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \; \frac{\pi r_0^2}{4} \left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2 e^{-\frac{1}{2} q^2 r_0^2}$$

$$q^2 = (\vec{k}' - \vec{k})^2 = 2k^2 - 2k^2 \cos\theta$$

$$\int_0^{\pi} \sin\theta \, d\theta \; e^{-k^2 r_0^2 (1 - \cos\theta)}$$

$$= e^{-k^2 r_0^2} \int_{-1}^{+1} dx \; e^{k^2 r_0^2 x}$$

$$= e^{-k^2 r_0^2} \frac{1}{k^2 r_0^2} \left(e^{k^2 r_0^2} - e^{-k^2 r_0^2} \right)$$

$$= \frac{1}{k^2 r_0^2} \left(1 - e^{-2k^2 r_0^2} \right) .$$

$$\therefore \quad \sigma = \frac{\pi^2}{2k^2}\left(\frac{\mu V_0 r_0^2}{\hbar^2}\right)^2\left(1 - e^{-2k^2 r_0^2}\right).$$

19.5.4

$$V(r) = - V_0 \, \Theta(r_0 - r)$$

For the s-wave ($l = 0$)

$$\psi = \psi(r) = A\, j_0(kr) + B\, \eta_0(kr)$$

$$= \frac{1}{kr}\left(A \sin kr - B \cos kr\right) \qquad \text{for } r > r_0$$

with $E = \dfrac{\hbar^2 k^2}{2\mu}$, and

$$\psi(r) = C\, j_0(k'r) = \frac{C}{k'r} \sin k'r \qquad \text{for } r < r_0$$

with $E = \dfrac{\hbar^2 k'^2}{2\mu} - V_0$.

$\psi(r)$ and $\dfrac{d\psi}{dr}$ must be continuous at $r = r_0$.

Hence

$$\frac{1}{k'} C \sin k' r_0 = \frac{1}{k}\left(A \sin k r_0 - B \cos k r_0\right)$$

and $$C \cos k' r_0 = A \cos k r_0 + B \sin k r_0$$

$$A = \left(\frac{k}{k'} \sin k' r_0 \sin k r_0 + \cos k' r_0 \cos k r_0 \right) C$$

$$B = \left(-\frac{k}{k'} \sin k' r_0 \cos k r_0 + \cos k' r_0 \sin k r_0 \right) C$$

The phase shift δ_ℓ is defined by

$$\psi_{\vec{k}}(\vec{r}) \underset{r \to \infty}{\longrightarrow} \sum_{\ell=0}^{\infty} A_\ell \frac{e^{i(kr - \ell\pi/2 + \delta_\ell)} - e^{-i(kr - \ell\frac{\pi}{2} + \delta_\ell)}}{r} P_\ell(\cos\theta)$$

In our case $(\ell = 0)$

$$\psi(r) = \frac{1}{ikr}\left[(A - iB) e^{ikr} - (A + iB) e^{-ikr} \right]$$

for $r > r_0$

Hence

$$e^{2i\delta_0} = \frac{A - iB}{A + iB} = \frac{\cos k' r_0 \, e^{-ikr_0} + i \frac{k}{k'} \sin k' r_0 \, e^{-ikr_0}}{\cos k' r_0 \, e^{ikr_0} - i \frac{k}{k'} \sin k' r_0 \, e^{ikr_0}}$$

$$= e^{-2ikr_0} \frac{\cos k' r_0 + i \frac{k}{k'} \sin k' r_0}{\cos k' r_0 - i \frac{k}{k'} \sin k' r_0}$$

$$= e^{-2ikr_0} \, e^{2i\beta}$$ where

$$\tan\beta = \frac{k}{k'}\tan k' r_0$$

$$\therefore \quad \delta_0 = -k r_0 + \tan^{-1}\left(\frac{k}{k'}\tan k' r_0\right)$$

For $$k' r_0 = \left(n+\tfrac{1}{2}\right)\pi + \underset{\substack{\uparrow \\ \text{small}}}{\epsilon}$$

$$\frac{k}{k'}\tan k' r_0 \simeq -\frac{k}{k'}\,\frac{1}{\epsilon}$$

$$\simeq \frac{-k}{k'\left[k' r_0 - \left(n+\tfrac{1}{2}\right)\pi\right]}$$

$$\simeq \frac{-\dfrac{k\hbar^2}{2\mu}\,\dfrac{1}{r_0}}{\dfrac{\hbar^2 k'^2}{2\mu} - V_0 + V_0 - \dfrac{\hbar^2 k'}{2\mu r_0}\left(n+\tfrac{1}{2}\right)\pi}$$

$$\simeq \frac{\Gamma/2}{V_0 - \dfrac{\hbar^2}{2\mu r_0^2}\left(n+\tfrac{1}{2}\right)^2\pi^2 - E}$$

with $$\Gamma = \frac{\hbar^2 k_n}{\mu r_0}$$

From Exercise 12.6.9, for bound states

$$k'/\chi_0 = -\tan k' r_0$$

where $\chi_0 = \frac{1}{\hbar}\sqrt{-2\mu E}$

For zero energy bound states, $\chi_0 \to 0$

$$\tan k'_m r_0 = \infty$$

$$\therefore \quad k'_m r_0 = \left(n + \tfrac{1}{2}\right)\pi$$

$$m = 0, 1, 2 \ldots$$

19.5.6

i.

$$\psi = e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r} \qquad \text{as } r \to \infty$$

$$\vec{j}^{\,inr} = \frac{\hbar}{2\mu i}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

$$= \frac{\hbar}{\mu}\,\mathrm{Im}\left[e^{-ikz}\,\vec{\nabla}\left(f(\theta)\frac{e^{ikr}}{r}\right) + f^*(\theta)\frac{e^{-ikr}}{r}\,\vec{\nabla} e^{ikz}\right]$$

$$j_r^{\,inr} = \frac{\hbar}{\mu}\,\mathrm{Im}\,\hat{r}\cdot\left[e^{-ikz} f(\theta)\frac{e^{ikr}}{r}\left(ik-\frac{1}{r}\right) + f^*(\theta)\frac{e^{-ikr+ikz}}{r}\,ik\hat{z}\right]$$

$$= \frac{\hbar k}{\mu r}\,\mathrm{Im}\left[e^{+ikr(1-\cos\theta)} f(\theta)\, i + f^*(\theta)\, i\, e^{ikr(\cos\theta-1)}\cos\theta\right] + O\left(\frac{1}{r^2}\right)$$

ii.

$$I_{\Delta\Omega} = \int_{\Delta\Omega} r^2\, d\Omega\; j_r^{\,int}$$

$$\Delta\Omega = \Delta\varphi \sin\theta\,\Delta\theta = \Delta\varphi\,\Delta\cos\theta = \Delta\varphi\,\Delta x$$

$$= \frac{\hbar^2 k}{\mu r}\,\mathrm{Im}\left\{ i f(\theta)\,\Delta\varphi \int_x^{x+\Delta x} dx\, e^{ikr(1-x)} + i f^*(\theta)\,\Delta\varphi \int_x^{x+\Delta x} dx\, e^{ikr(x-1)}\, x \right\}$$

$$I^{int}_{\Delta\Omega} = \frac{\hbar \Delta\varphi}{\mu} \,\Im m \Big\{ f(\theta)\Big[e^{ikr(1-\cos(\theta+\Delta\theta))} - e^{ikr(1-\cos\theta)}\Big]$$

$$- f^*(\theta)\cos\theta \Big[e^{-ikr(1-\cos(\theta+\Delta\theta))} - e^{-ikr(1-\cos\theta)} \Big]\Big\}$$

For $\theta \neq 0$ and $\Delta\theta$ small,

$I^{int}_{\Delta\Omega}$ averages to zero as $r \to \infty$.

$\underline{iii}$. Let $\theta = 0$, $\Delta\theta$ small and $\Delta\varphi = 2\pi$.

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = \frac{\hbar\, 2\pi}{\mu} \,\Im m \Big\{ f(0)\Big[e^{ikr(1-\cos\Delta\theta)} - 1\Big]$$

$$- f^*(0)\Big[e^{-ikr(1-\cos\Delta\theta)} - 1\Big]\Big\}$$

The exponentials average to zero as $r \to \infty$.

Hence

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = -\frac{\hbar\, 2\pi}{\mu} \,\Im m \left(f(0) - f^*(0)\right)$$

$$I^{int}_{\substack{\Delta\Omega \\ \text{forward}}} = -\frac{4\pi\hbar}{\mu} \operatorname{Im} f(0)$$

Note that the RHS does not depend upon $\Delta\Omega = \pi(\Delta\theta)^2$.

$$\vec{j}^{\,tot} = \frac{\hbar}{\mu} \operatorname{Im}\left[(\psi^{inc} + \psi^{scatt})^* \vec{\nabla}(\psi^{inc} + \psi^{scatt}) \right]$$

$$= \frac{\hbar k}{\mu}\hat{z} + \vec{j}^{\,int} + \frac{\hbar}{\mu} \operatorname{Im}\left(\psi^{scatt\,*} \vec{\nabla} \psi^{scatt} \right)$$

$$I^{tot}_{\substack{\Delta\Omega \\ \text{forward}}} = \frac{\hbar k}{\mu} r^2 \Delta\Omega + I^{int}_{\substack{\Delta\Omega \\ \text{forward}}}$$

$$+ \frac{\hbar}{\mu} |f(0)|^2 k \, \Delta\Omega$$

Take the limit

$$r \to \infty \quad , \quad \Delta\Omega \to 0$$

$$\Delta S = r^2 \Delta\Omega \quad \text{finite}$$

The last term drops out.

In the absence of scatterer

$$I^{int}_{\substack{\Delta S \\ \text{forward}}} = \frac{\hbar k}{\mu} \Delta S = (\text{incident flux}) \times \Delta S$$

When the scatterer is present

$$I^{int}_{\substack{\Delta S \\ \text{forward}}} = \frac{\hbar k}{\mu} \Delta S - \frac{4\pi \hbar}{\mu} \operatorname{Im} f(0)$$

Hence

$$\frac{4\pi \hbar}{\mu} \operatorname{Im} f(0) = \#\ \text{particles scattered per unit time}$$

$$= \sigma_{tot} \times \text{incident flux}$$

$$= \sigma_{tot} \frac{\hbar k}{\mu}$$

Hence

$$\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(0) .$$