Lecture 39

Friday, April 16, 2021

BCS $\Rightarrow$ ATTRACTIVE INTERACTION

( PHONONS $\rightarrow$ RETARDED INTERACTION )

COOPER PROBLEM (INSTABILITY)

$\vec{k}\uparrow, \; -\vec{k}\downarrow$

$$\chi(\vec{r}) = \int \frac{d\vec{k}}{(2\pi)^3}\, g(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}$$

$$(H_0 + H_1 - \epsilon)\,\chi(r) = 0$$

$\uparrow$ ATTRACTIVE INTERACTION

$$(E_k - \epsilon)\, g(\vec{k}) + \int \frac{d\vec{k}'}{(2\pi)^3} \langle k | H_1 | k' \rangle\, g(\vec{k}') = 0$$

NONINTERACTING
ENERGY

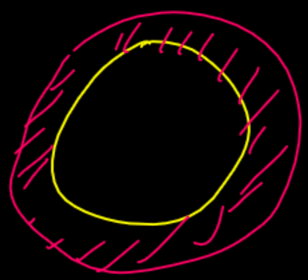
CHANGE TO ENERGY $\longrightarrow$ $\vec{k} \to E$

$$\underbrace{}_{}(E - \epsilon)\, g(E) + \int dE' \underbrace{N(E')}_{} H_1(E,E')\, g(E') = 0$$

DENSITY OF
STATES IN ENERGY
(2 PARTICLE, WITH $\vec{K}=0$)

SIMPLIFIED $H_1$ :

(ATTRACTIVE) $$H_1(E,E') \equiv \begin{cases} -V & E_F \leq E, E' \leq E_F + \hbar\omega_D \\ 0 & \text{OTHERWISE} \end{cases}$$



THIN STRIP
WIDTH $\hbar\omega_D$
ABOUT $E_F$

$$(E-\epsilon)\, g(E) = + \underbrace{V \int_{2E_F}^{2(E_F+\hbar\omega_D)} dE'\, N(E')\, g(E')}_{C}$$

"2" ELECTRONS

$C$ = CONSTANT (IND. OF $E$)

$$g(E) = \frac{C}{E-\epsilon}$$

$E - \epsilon$

$$\boxed{1 = V \int_{2\epsilon_F}^{2(\epsilon_F + \hbar\omega_D)} \frac{dE'\, N(E')}{E' - \epsilon}}$$

WHAT IS $\epsilon$?

① NO MATTER HOW SMALL $V$ IS, THERE ALWAYS IS A SOLUTION WITH $\epsilon < 2\epsilon_F$

PROVIDED $N(\epsilon_F) \neq 0$

$\Rightarrow$ 3D WILL ALWAYS HAVE A BOUND STATE!!

$N(E) \approx N(E_F) \sim$ CONSTANT

$$1 = V N(E_F) \underbrace{\int_{2\varepsilon_F}^{2(\varepsilon_F + \hbar\omega_D)} \frac{1}{E' - \varepsilon} \, dE'}$$

$$\ln\left( E' - \varepsilon \right) \Big|_{2\varepsilon_F}^{2(\varepsilon_F + \hbar\omega_D)}$$

$$1 = V N(E_F) \ln\left( \frac{\Delta + 2\hbar\omega_D}{\Delta} \right)$$

$$\varepsilon = 2E_F - \Delta$$

$\Downarrow$

BINDING ENERGY OF PAIR

$$\Delta = \frac{2\hbar\omega_D}{\exp\left(\frac{1}{N(E_F)V}\right) - 1} \approx \boxed{2\hbar\omega_D \exp\left(-\frac{1}{N(E_F)V}\right)}$$

1. FOR $V > 0 \Rightarrow$ BOUND STATES EXIST $(-V)$
2. NONANALYTIC IN $V$
3. SIGNIFIES INSTABILITY OF FERMI SEA TO WEAK ATTRACTIVE INTERACTION.

BCS GROUND STATE $(\epsilon)$

$\Rightarrow$ COOPER PAIRS $\quad k\uparrow, -k\downarrow$

$\Rightarrow$ { "COHERENCE LENGTH" ( SPATIAL EXTENT OF COOPER PAIRS)

PAIRS ACT LIKE "BOSONS"

## BCS HAMILTONIAN

$k, -k$ HAVE OPP. SPINS

$$H_{RED} = \sum_{k} E_k \left( C_k^+ C_k + C_{-k}^+ C_{-k} \right)$$

CREATION & ANNIHALATION

$$- V \sum_{k k'} C_{k'}^+ C_{-k'}^+ C_{-k} C_k$$

MANY BODY HAMILTONIAN

(A.K.A "HARD

$$\underbrace{\quad k\,k' \quad}_{\text{INTERACTION TERM}}$$

(A.K.A "HARD PART")

"MEAN FIELD THEORY"

$$\Delta_k = V \sum_{k'} \langle C_{-k'\downarrow} C_{k'\uparrow} \rangle$$

$$H_{BCS} = \sum_k \epsilon_k \left( C_k^{+} C_k + C_{-k}^{+} C_k \right)$$

(MEAN FIELD) THEORY

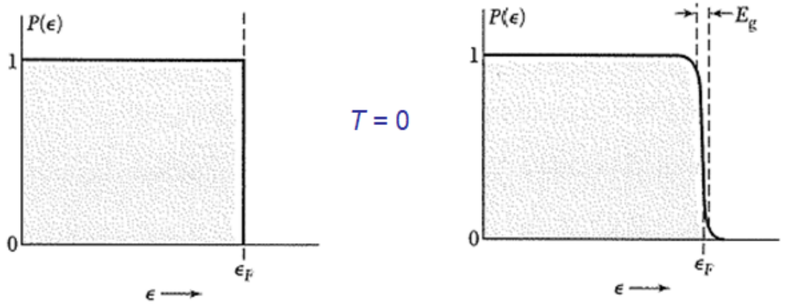
$$- \sum_k \Delta_k C_k^{+} C_{-k}^{+} - \sum_k \Delta_k^{*} C_{-k} C_k$$

DIAGONALIZED EXACTLY (BOGOLIUBOV - VALATIN TRANSFORMATION)

$$\mathcal{E}_p = \pm\sqrt{E_p^2 + |\Delta_p|^2}$$

INTERACTING — NORMAL — GAP

## BCS Ground State




Cooper pair: 1-e occupancy with $T_{eff} = T_C$

Normal State

Super Conducting State: Cooper pair mixes e's from below & above $\varepsilon_F$

$(K.E.)_{normal} < (K.E.)_{super}$ but $E_{normal} > E_{super}$ due to $-U$.

Cooper pair: $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow) \rightarrow$ spin = 0 (boson)

S.C. HAS HIGHER KINETIC ENERGY THAN NORMAL STATE.

BUT TOTAL ENERGY IS LOWER (DUE TO POTENTIAL)