Midterm Exam 2 – PHZ 4404 Spring 2021
SOLID STATE PHYSICS
Friday, April 16, 2021
Due: Wednesday, April 21, 2021 at 8PM uploaded to the canvas site.

This is a take-home exam. It is due on Wednesday, April 21, 2021 ***at 8pm.*** When you are finished, please upload to the Canvas Assignments Page under Midterm 2. If you have problems, you can email me and include it as an attachment. You are allowed to use your textbook "Introduction to Solid State Physics" by Kittel, your class notes only (and of course a calculator) and anything that is on the main WEB page (including links and notes) but not any other WEB pages. You can discuss the questions only with the instructor. Please email me if you would like to set up a Zoom meeting. You **are not allowed** to work with anyone else. Your name and signature above indicates that you have obeyed the honor system on this test and have not *received* or *given* aid to or from anyone on this test.

Show All Your Work!!!!

- Label all additional work with your name in the upper right hand corner of any additional sheets you turn in.
- All work (on problems) must be shown to receive full credit.
- All work must be **clear** and **unambiguous** to receive full credit. Cross out any work that you do not want counted. If you have the right answer, but wrong steps, you will not receive full credit.
- All units must be shown to receive full credit.
- *Indicate the final answer by **boxing** the result.*

1. ***Electrons in Semiconductors and Density of States.*** **(25 points)** As stated in class, a better (than simple effective mass theory) description for the energy levels of an electron in a semiconductor is given by the ε vs. **k** relationship:

$$\varepsilon(1+\alpha\varepsilon)=\frac{\hbar^2 k^2}{2m^*}$$

where m^* is the effective mass (usually different from m_0) *at the band minimum* and α is a parameter dealing with the *nonparabolicity* of the electron band.

   a. Solve for $\varepsilon(\mathbf{k})$.
   b. How does the electron energy ε depend on wave vector **k** for $\alpha\varepsilon \ll 1$?, for $\alpha\varepsilon \gg 1$?
   c. As stated in class, the velocity of an electron in a crystal can be shown to be related to the ε vs. **k** relationship through: $\mathbf{v}=\frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon(\mathbf{k})$, where $\nabla_{\mathbf{k}}$ is the gradient with respect to the wavevector **k**. What is the electron velocity (in terms of **k** etc.) for $\alpha\varepsilon \ll 1$?, for $\alpha\varepsilon \gg 1$?
   d. What is the energy density of states (per unit volume) (***for all values of*** ε not just the limiting cases) in three dimensions?, in two dimensions? Be sure to give your final answer in terms of ε and not **k.**

2. ***Landau Levels in the free electron gas.*** (**25 points**) (Kittel, Problem 9.11) In appendix G, it is shown that the quantum mechanical Hamiltonian, which includes a magnetic field, is given by:

$$H=\frac{1}{2m}\left(\frac{\hbar\nabla}{i}+\frac{e}{c}\mathbf{A}\right)^2$$

where **A** is the vector potential with $\mathbf{B}=\nabla\times\mathbf{A}$ and (-e) is the charge of the electron. (We do not worry about external potentials, including the crystal potential in this problem). The vector potential **A** can be chosen in different ways to yield the same magnetic field (this is called a Gauge Transformation or choosing a gauge). For a uniform magnetic field, $\mathbf{B}=B\hat{\mathbf{z}}$, one can take $\mathbf{A}=-By\ \hat{\mathbf{x}}$ which is known as the ***Landau Gauge***. The Hamiltonian of a free electron without spin is therefore

$$H\ =\ -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}-\frac{eyB}{c}\right)^2$$

Look for an eigenfunction of the wave equation $H\psi=\varepsilon\psi$ in the form

$$\psi(x,y,z)=\chi(y)\ \exp\left[i\left(k_x x+k_z z\right)\right].$$

a. Show that $\chi(y)$ satisfies the equation

$$\frac{\hbar^2}{2m}\frac{\partial^2\chi(y)}{\partial y^2}+\left[\varepsilon-\frac{\hbar^2 k_z^{\ 2}}{2m}-\frac{m\omega_c^2\left(y-y_0\right)^2}{2}\right]\chi(y)=0,$$

where $\omega_c = eB/mc$ and $y_0 = -c\hbar k_x / eB$.

b. Show that this is the wave equation of a harmonic oscillator with frequency ω_c , where

$$\varepsilon_n = \left(n + \frac{1}{2}\right)\hbar\omega_c + \frac{\hbar^2 k_z^{\,2}}{2m}.$$

Note that this problem can be done in a different gauge, $\mathbf{A} = -\frac{By}{2}\,\hat{\mathbf{x}} + \frac{Bx}{2}\,\hat{\mathbf{y}}$ which is known as the ***symmetric*** gauge. It is harder to do in this gauge, but interesting non-the-less.

3. ***Phonons with Damping***. **(25 points)** We have calculated 1D phonon modes with the force coming from nearest neighbor atoms on the lattice (monatomic chain). If a dissipative term, $-M\gamma\dot{u}_n$, where M is the atomic mass and u_n is the displacement of the atom on site *n*, is added to the force, how will the solution change?

   (a) Write down the equation of motion for u_n.

   (b) Assume that the solution has the form $u_n = e^{i(qna-\omega t)}$ find ω as a function of qa, M, γ and the spring constant *K* between the neighboring atoms.

   (c) What is the lifetime of the phonon?

4. ***Anharmonic Corrections to the Classical Specific Heat.*** (**25 points**) Consider a classical harmonic oscillator described by a single particle of mass *m* in a potential given by:

$$U(x) = U_H(x) + U_A(x)$$

where

$$U_H(x) = a_2\, x^2$$
$$U_A(x) = a_3\, x^3 + a_4\, x^4$$

with a_2, a_3, a_4 are constants. Show that the anharmonic contributions lead to a correction to the classical (high temperature) specific heat which is linear in T. Why must the quartic term be included?

Hint: The classical partition function,

$$Z = \iint dp\ dx\ \exp(-E(p,x)/k_B T)$$

can be calculated without difficulty if the anharmonic contribution U_A is small compared to U_H so one can expand an appropriate part of the exponential (***comment on this***). The average energy is related to the partition function through

$$\langle E \rangle = k_B T^2 \frac{d(\ln Z)}{dt},$$

and one should use the expansion of $\ln(1+x)$ for small x.