Chapter 22 - Coulomb's Law

The magnitude of the electrostatic force between electric point charges q_1 and q_2 separated by a distance r is

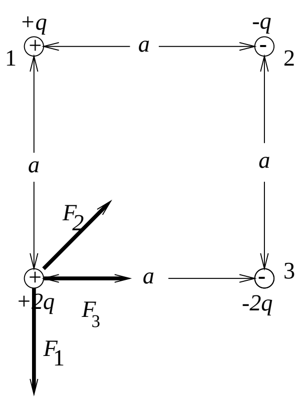
$$F = \frac{K|q_1||q_2|}{r^2}.$$

The force is directed along the line joining the charges. If the charges have the same sign, the force is repulsive. If the charges have opposite signs, the force is attractive.

The principle of superposition holds. The force due to a set of point charges acting on another charge is the vector sum of the forces due to each charge in the set as if it were acting alone:

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots \\ &= (F_{1x} + F_{2x} + F_{3x} + \ldots)\hat{x} + (F_{1y} + F_{2y} + F_{3y} + \ldots)\hat{y}. \end{aligned}$$

Example 1 (22-10P): What are the horizontal and vertical components of the resultant electrostatic force on the charge in the lower left corner of the square if $q = 1.0 \times 10^{-7}$ C and $a = 5.0$ cm?



Procedure:

1. Draw in the forces acting on the charge ($2q$) in the lower left corner.
2. Write the vector equation and take components.

$$\begin{aligned} \mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= -F_1\hat{y} + F_2\cos 45^\circ\hat{x} + F_2\sin 45^\circ\hat{y} + F_3\hat{x} \\ &= (F_2\cos 45^\circ + F_3)\hat{x} + (-F_1 + F_2\sin 45^\circ)\hat{y} \end{aligned}$$

If the system is in equilibrium, $\mathbf{F} = 0$.

3. Calculate the forces using Coulomb's law. (Ignore the signs on the charges.)

$$F_1 = \frac{K|q||2q|}{a^2} = \frac{K2q^2}{a^2} = 0.072 \text{ N}$$
$$F_2 = \frac{K|-q||2q|}{2a^2} = \frac{K2q^2}{2a^2} = 0.036 \text{ N}$$
$$F_3 = \frac{K|2q||-2q|}{a^2} = \frac{K4q^2}{a^2} = 0.144 \text{ N}$$

Note - Sometimes we have to leave the forces in symbolic form.

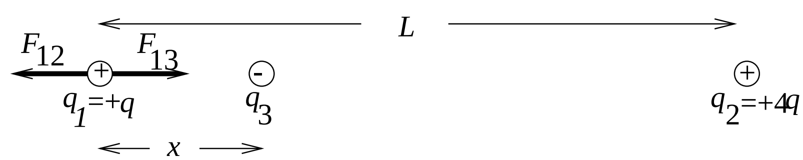
4. Calculate **F**.

$$\begin{aligned}\mathbf{F} &= (F_2 \cos 45^\circ + F_3)\hat{x} + (-F_1 + F_2 \sin 45^\circ)\hat{y} \\ &= (0.036 \text{ N} \cos 45^\circ + 0.144 \text{ N})\hat{x} + (-0.072 \text{ N} + 0.036 \text{ N} \sin 45^\circ)\hat{y} \\ &= 0.138 \text{ N}\hat{x} - 0.047 \text{ N}\hat{y}\end{aligned}$$

The magnitude of F would be:

$$F = \sqrt{(F_x)^2 + (F_y)^2} = \sqrt{(0.138 \text{ N})^2 + (-0.047 \text{ N})^2} = 0.146 \text{ N}.$$

Example 2 (22-16P): Two *free* point charges $+q$ and $+4q$ are a distance L apart. A third charge is placed so that the entire system is in equilibrium. Find the location, magnitude, and sign of the third charge.



The procedure is the same as before. However, step 1 is the hardest. We do not know the sign or location of q_3 so we have to try all possible combinations.

After some trial and error, we find that q_3 must be negative and between q_1 and q_2.

1. Draw in the forces acting on q_1.
2. Write the vector equation and take components.

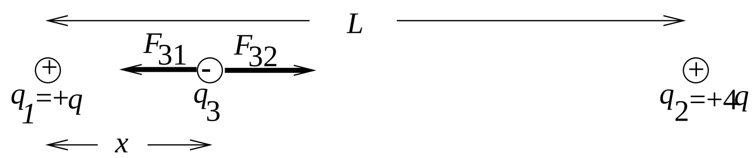
$$\begin{aligned}\mathbf{F}_1 &= \mathbf{F}_{12} + \mathbf{F}_{13} \\ &= -F_{12}\hat{x} + F_{13}\hat{x} \\ &= (F_{13} - F_{12})\hat{x}.\end{aligned}$$

3. Set $\mathbf{F}_1 = 0$ and solve.
   Since $\mathbf{F}_1 = 0$, we have $F_{13} = F_{12}$. This gives

$$\begin{aligned} F_{13} &= F_{12} \\ \frac{K|q_1||q_3|}{x^2} &= \frac{K|q_1||q_2|}{L^2} \\ \frac{|q_3|}{x^2} &= \frac{|q_2|}{L^2}. \end{aligned}$$

   We cannot go any further since there are two unknowns x and q_3.

Now we consider the forces on q_3.



1. Draw in the forces acting on q_3.
2. Write the vector equation and take components.

$$\begin{aligned} \mathbf{F}_3 &= \mathbf{F}_{31} + \mathbf{F}_{32} \\ &= -F_{31}\hat{x} + F_{32}\hat{x} \\ &= (F_{32} - F_{31})\hat{x}. \end{aligned}$$

3. Set $\mathbf{F}_3 = 0$ and solve.
   Again $\mathbf{F}_3 = 0$, we have $F_{32} = F_{31}$. This gives

$$\begin{aligned} F_{32} &= F_{31} \\ \frac{K|q_3||q_2|}{(L-x)^2} &= \frac{K|q_3||q_1|}{x^2} \\ \frac{|q_2|}{(L-x)^2} &= \frac{|q_1|}{x^2} \\ \frac{4q}{(L-x)^2} &= \frac{q}{x^2} \\ 4x^2 &= (L-x)^2 \\ &= L^2 - 2Lx + x^2 \\ 0 &= 3x^2 + 2Lx - L^2 \\ &= (3x - L)(x + L). \end{aligned}$$

The only physically realistic solution is $x = L/3$.

Returning to find the magnitude of q_3 we have

$$\begin{aligned}
\frac{|q_3|}{x^2} &= \frac{|q_2|}{L^2} \\
\frac{|q_3|}{(L/3)^2} &= \frac{4q}{L^2} \\
|q_3| &= \frac{4q}{9}
\end{aligned}$$

and we know the sign of q_3 so $q_3 = -4q/9$.

Chapter 23 - Electric fields

One way to explain the electrostatic force between charges is to assume that each charge sets up an electric field in the space around it. The electrostatic force acting on any one charge is then due to the electric field set up at its location by the remaining charges.

The *electric field* $\mathbf{E}$ at any point is defined in terms of the electrostatic force $\mathbf{F}$ that would be exerted on a positive test charqe q_0 placed there:

$$\mathbf{E} = \frac{\mathbf{F}}{q_0}.$$

The force that an electric field exerts on a negative charge is in the opposite direction to the force on a positive charge.

The magnitude of the electric field a distance r away from a point charge q is

$$E = \frac{K|q|}{r^2}$$

The direction of $\mathbf{E}$ is given by the convention that electric fields orginate on positive charges and terminate on negative charges.

Again the principle of superposition holds. If there are many charges contributing to an electric field, the net electric field is just the vector sum of the electric fields due to each of the charges.

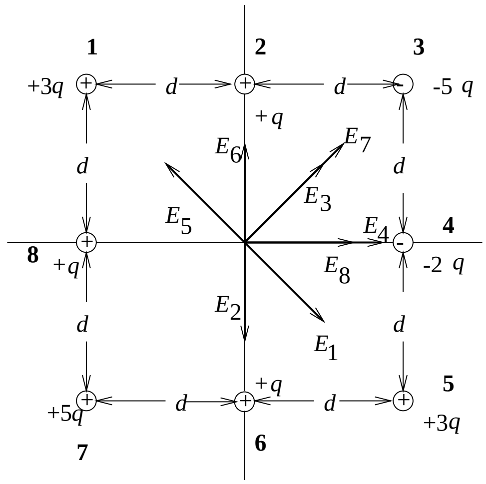
The electric field due to a *continuous charge distribution* is found by treating the charge elements as point charges and then summing, via integration, the electric field vectors produced by all the charge elements.

Electric field lines provide a means for visualizing the direction and magnitude of electric fields. The electric field vector is tangent to the a field line through that point.

- Field lines originate on positive charges and terminate on negative charges.
- The number of field lines leaving or entering a charge is proportional to the magnitude of the charge.
- The density of field lines in any region is proportional to the magnitude of the electric field in that region.

Example 1 (23-21P): Four charges form the corners of a square and four more charges lie at the midpoints of the sides of the square. The distance between adjacent charges on the perimeter of the square is d. What are the magnitude and direction of the electric field at the center of the square?

Procedure: (Very similar to the plan used in the previous chapter)



1. Draw the electric field acting at the center of the square.

2. Write the vector equation and take components.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E_3} + \mathbf{E_4} + \mathbf{E}_5 + \mathbf{E_6} + \mathbf{E}_7 + \mathbf{E}_8.$$

We could take the components of each of the 8 electric fields. However some of the field will cancel. The field due to the $+q$s and the $+3q$s cancel. There is only 4 fields to consider.

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_3 + \mathbf{E}_4 + \mathbf{E}_7 + \mathbf{E}_8 \\
&= (E_3 \cos 45^\circ \hat{x} + E_3 \sin 45^\circ \hat{y}) + E_4 \hat{x} + (E_7 \cos 45^\circ \hat{x} + E_7 \sin 45^\circ \hat{y}) + E_8 \hat{x} \\
&= (E_3 \cos 45^\circ + E_4 + E_7 \cos 45^\circ + E_8)\hat{x} + (E_3 \sin 45^\circ + E_7 \sin 45^\circ)\hat{y}
\end{aligned}$$

3. Calculate the magnitude of the electric fields.

$$\begin{aligned}
E_3 &= \frac{K|-5q|}{2d^2} = \frac{5q}{2d^2} \\
E_4 &= \frac{K|-2q|}{d^2} = \frac{2q}{d^2} \\
E_7 &= \frac{K|5q|}{2d^2} = \frac{K5q}{2d^2} \\
E_8 &= \frac{K|q|}{d^2} = \frac{Kq}{d^2}
\end{aligned}$$

4. Calculate the $\mathbf{E}$.

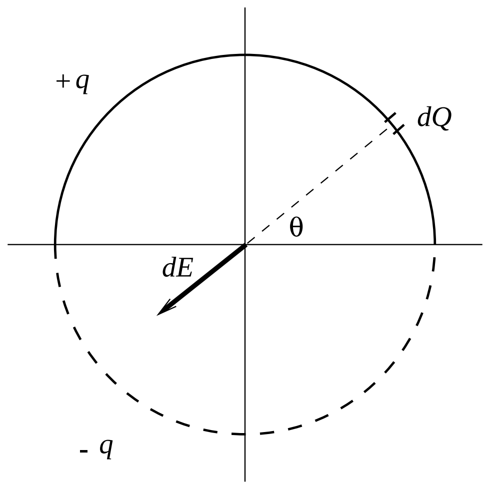
$$\mathbf{E} = (E_3 \cos 45^\circ + E_4 + E_7 \cos 45^\circ + E_8)\hat{x} + (E_3 \sin 45^\circ + E_7 \sin 45^\circ)\hat{y}$$

$$
\begin{aligned}
&= \left(\frac{5q}{2d^2}\frac{1}{\sqrt{2}} + \frac{2q}{d^2} + \frac{K5q}{2d^2}\frac{1}{\sqrt{2}} + \frac{Kq}{d^2}\right)\hat{x} + \left(\frac{5q}{2d^2}\frac{1}{\sqrt{2}} + \frac{K5q}{2d^2}\frac{1}{\sqrt{2}}\right)\hat{y} \\
&= \frac{Kq}{d^2}\left(\frac{5}{2\sqrt{2}} + 2 + \frac{5}{2\sqrt{2}} + 1\right)\hat{x} + \frac{Kq}{d^2}\left(\frac{5}{2\sqrt{2}} + \frac{5}{2\sqrt{2}}\right)\hat{y} \\
&= \frac{Kq}{d^2}\left(3 + \frac{5}{\sqrt{2}}\right)\hat{x} + \frac{Kq}{d^2}\left(\frac{5}{\sqrt{2}}\right)\hat{y} \\
&= \frac{Kq}{d^2}(6.536)\,\hat{x} + \frac{Kq}{d^2}(3.536)\,\hat{y}.
\end{aligned}
$$

The magnitude will be $E = (\sqrt{6.536^2 + 3.536^2})Kq/d^2 = 7.430Kq/d^2$ and its direction is $\theta = \arctan E_y/E_x = 28.4°$.

Example 2 (23-31P): Two curved plastic rods, one of charge $+q$ and the other of charge $-q$, form a circle of radius R in an xy plane. The x axis passes through their connecting points, and the charge is distributed uniformly on both rods. What are the magnitude and direction of the electric field $\mathbf{E}$ produced at P, the center of the circle?

Procedure: (See pages 563-564 *Problem Solving Tactics*)The general stategy is to pick out an



element dq of the charge, find $d\mathbf{E}$ due to that element, and integrate $d\mathbf{E}$ over the entire line of charge.

1. If the line of charge is circular, let ds be the arc length of an element of the distribution. If the line is straight, run an x axis along it and let dx be the length of an element. Mark the element on the sketch.
2. Relate the charge dq of the element to the length of the element with either $dq = \lambda\, ds$ or $dq = \lambda\, dx$. Consider dq and λ to be positive, even if the charge is negative.

3. Express the field $d\mathbf{E}$ produced by dq at point P.

$$\begin{aligned} dE &= \frac{K\,dQ}{r^2} \\ &= \frac{K\lambda\,ds}{R^2} \\ &= \frac{K\lambda R\,d\theta}{R^2} \\ &= \frac{K\lambda\,d\theta}{R}. \end{aligned}$$

4. Look for any symmetries.
The x components will cancel. Since the magnitudes of the charges on the upper and lower semicircles are the same, we can find the electric field due to the top semicircle and double it to find the total field. Therefore,

$$dE_y = -2\frac{K\lambda\,d\theta}{R}\sin\theta.$$

5. Evaluate the integral.

$$\begin{aligned} E_y &= -2\frac{K\lambda}{R}\int_0^{\pi}\sin\theta\,d\theta \\ &= -2\frac{K\lambda}{R}\left[-\cos\theta\right]_0^{\pi} \\ &= -2\frac{K\lambda}{R}2 \\ &= -\frac{4K\lambda}{R}. \end{aligned}$$

6. If λ is not given, then it must be eliminated.

$$\lambda = \frac{\text{charge}}{\text{length}} = \frac{q}{\pi R}.$$

Finally, the magnitude of the electric field is (take absolute value)

$$E = \frac{4Kq}{\pi R^2}.$$

Chapter 24 - Gauss' law

The electric flux through a surface is given by

$$\Phi = \int \mathbf{E} \cdot d\mathbf{A}.$$

(The element of area vector $d\mathbf{A}$ is directed perpendicular to the area of interest.) The flux can be positive, negative, or zero depending upon the angle between $\mathbf{E}$ and $d\mathbf{A}$.

- If the angle between $\mathbf{E}$ and $d\mathbf{A}$ is between 0 and $\pi/2$ then the flux is positive.
- If the angle between $\mathbf{E}$ and $d\mathbf{A}$ is $\pi/2$ then the flux is zero.
- If the angle between $\mathbf{E}$ and $d\mathbf{A}$ is between $\pi/2$ and π then the flux is negative.

When the area encloses an volume, the element of area is directed normal to the surface and away from the enclosed volume. If the real (or imaginary) surface surrounds a charge, there will be a net flux through the surface. If the charge is external to the enclosed volume, there is no net flux from that charge.

Gauss' law is

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{enclosed}}{\epsilon_0}$$

in which $q_{enclosed}$ is the net charge inside an imaginary closed surface (a *Gaussian surface*).

Some facts about conductors in electrostatic conditions.

- The electric field inside a conductor vanishes (or the electrostatic forces would prevent static conditions).
- Any excess charge resides on the surface of the conductor (from above and Gauss' law).
- The electric field enters a conductor perpendicular to the surface (or again there will not be static conditions because of the tangential component of the electric field.)
- A conductor shields the electric field from a charge external to the conductor but the electric field from an internal charge is unshielded.

There are three applications of Gauss' law.

1. Calculate the electric flux from a given charge distribution.
2. Calculate the electric field from a given symmetric charge distribution. Here the symmetry makes the form of the electric field known.
3. Calculate the charge distribution for a given electric field.

Example 1 (24-52P): A solid nonconducting sphere of radius R has a nonuniform charge distribution of volume charge density $\rho = \rho_s r/R$, where ρ_s is a constant and r is the distance from the center of the sphere. Show that (a) the total charge on the sphere is $Q = \pi \rho_s R^3$ and (b) the electric field inside the sphere has a magnitude given by

$$E = \frac{KQ}{R^4} r^2.$$

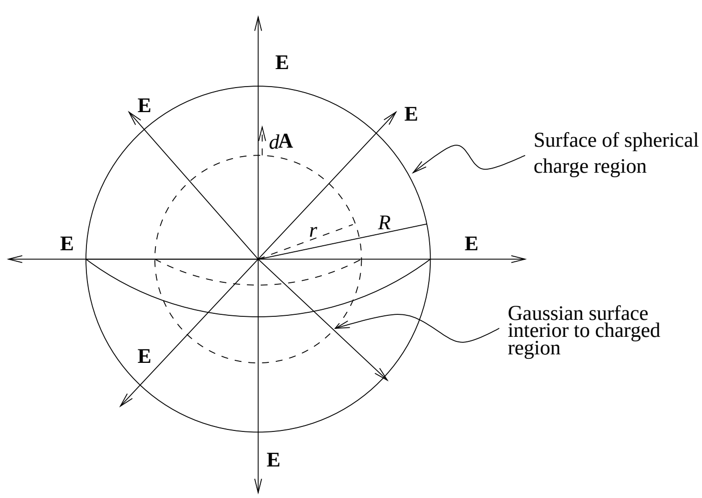
Procedure:

1. Part (a) is just a straightforward integration:

$$\begin{aligned} Q &= \int \rho \, dV \\ &= \int \frac{\rho_s r}{R} 4\pi r^2 dr \\ &= \frac{4\pi\rho_s}{R} \int_0^R r^3 dr \\ &= \frac{4\pi\rho_s}{R} \frac{R^4}{4} \\ &= \pi \rho_s R^3. \end{aligned}$$

   Which gives $\rho_s = Q/\pi R^3$.

2. Use Gauss' law to solve part (b). Take a Gaussian surface at a radius r concentric with the sphere and interior to it.



3. Evaluate the flux through the Gaussian surface.

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{A} &= \oint E\, dA \\
&= E \oint dA \\
&= EA \\
&= E4\pi r^2.
\end{aligned}$$

4. Evaluate the charge enclosed (in the Gaussian surface).

$$\begin{aligned}
q_{enclosed} &= \int \rho dV \\
&= \int \frac{\rho_s r}{R} 4\pi r^2 \\
&= \frac{4\pi\rho_s}{R} \int_0^r r^3 dr \\
&= \frac{4\pi\rho_s}{R} \frac{r^4}{4} \\
&= \frac{\pi\rho_s r^4}{R} \\
&= \frac{\pi}{R} \frac{Q}{\pi R^3} r^4 \\
&= \frac{Qr^4}{R^4}.
\end{aligned}$$

5. Substitute into Gauss' law and solve.

$$\begin{aligned}
\oint \mathbf{E} \cdot d\mathbf{A} &= \frac{q_{enclosed}}{\epsilon_0} \\
4\pi r^2 E &= \frac{Qr^4}{\epsilon_0 R^4} \\
E &= \frac{1}{4\pi\epsilon_0} \frac{Qr^2}{R^4} \\
&= \frac{KQr^2}{R^4}.
\end{aligned}$$