**Test 1 solutions**

**Phz 3113 Fall 2007**

1. Expand $x/(e^x - 1)$ to order $x^2$ for $x \ll 1$.

$$\frac{x}{1 + x + x^2/2! + x^3/3! + \cdots - 1} \tag{1}$$

To order $x^2$,

$$\frac{1}{(1 + x/2! + x^2/3! + \ldots)} \simeq \frac{1}{1 + (x/2! + x^2/3!)}, \tag{2}$$

which is an alternating geometric series in $(x/2! + x^2/3!)$, so

$$= 1 - (x/2! + x^2/3!) + (x/2! + x^2/3!)^2 \simeq 1 - x/2 + x^2/12 \tag{3}$$

2. The equation of state for a van der Waals gas is

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT, \tag{4}$$

where $a, b$ and $R$ are constants. Consider two experiments on such a gas confined to a cylinder where you may control $p, V$ and/or $T$.

(a) Hold $T$ constant and find $dV/dp$.

$$d\left[\left(p + \frac{a}{V^2}\right)(V - b)\right] = d(RT) = 0 \tag{5}$$

so

$$\frac{dV}{dp} = \frac{b - V}{\frac{a}{V^2} + \frac{2(b-V)a}{V^3} + p} = \frac{(b - V)V^3}{pV^3 - aV + 2ab} \tag{6}$$

(b) Hold $p$ constant and find $dV/dT$.

Similarly if $dp = 0$,

$$\frac{dV}{dT} = \frac{R}{\frac{a}{V^2} + \frac{2(b-V)a}{V^3} + p} = \frac{RV^3}{pV^3 - aV + 2ab} \tag{7}$$

3. Change variables $x = u + v$, $y = u - v$, to rewrite the differential equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 1 \tag{8}$$

in terms of $u$ and $v$ (no need to solve the equation).

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Sketch solution. First invert $u = (x+y)/2$, $v = (x-y)/2$. calculate partial derivatives:

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = \frac{1}{2}\left[\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}\right] \\ &\Rightarrow \frac{\partial}{\partial x} = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \end{aligned} \tag{9}$$

and similarly for y

$$\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \tag{10}$$

The 2nd partials are e.g.

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{1}{4}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) w \\ &= \frac{1}{4}\left[\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} + 2\frac{\partial^2 w}{\partial u \partial v}\right] \end{aligned} \tag{11}$$

and similarly for $y$ except the coefficient of the mixed partial derivative is negative. Constructing $\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 1$, we find

$$\frac{\partial^2 w}{\partial u \partial v} = 1. \tag{12}$$

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4. Evaluate the integral

$$\int_{y=0}^{\pi} dy \int_{x=y}^{\pi} dx \frac{\sin x}{x}. \tag{13}$$

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Reverse order of integrations:

$$\int_0^{\pi} dx \frac{\sin x}{x} \int_0^x dy = \int_0^{\pi} dx \sin x = -\cos x|_0^{\pi} = 2. \tag{14}$$

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5. If $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$, show that

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}. \tag{15}$$

[Hint: $\epsilon_{ijk}\epsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}$]

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$$\begin{aligned}
[\vec{\nabla}\times(\vec{A}\times\vec{B})]_i &= \epsilon_{ijk}\nabla_j(\vec{A}\times\vec{B})_k = \epsilon_{ijk}\epsilon_{kmn}A_mB_n = \epsilon_{kij}\epsilon_{kmn}\nabla_jA_mB_n \\
&= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\nabla_jA_mB_n = \nabla_jA_iB_j - \nabla_jA_jB_i \\
&= A_i(\vec{\nabla}\cdot B) + (\vec{B}\cdot\vec{\nabla})A_i - B_i(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})B_i = (\vec{B}\cdot\vec{\nabla})A_i - (\vec{A}\cdot\vec{\nabla})B_i,
\end{aligned}$$

where in the last step I used the info given that both vectors had zero divergence, and the product rule of differentiation.

6. Look for a minimum of the function $1/x + 4/y + 9/z$ for $x, y, z > 0$ and $x + y + z = 12$.

Lagrange multipliers:

$$F = 1/x + 4/y + 9/z + \lambda(x + y + z - 12), \tag{16}$$

so minimize:

$$\frac{\partial F}{\partial x} = -\frac{1}{x^2} + \lambda = 0 \;\; ; \;\; \frac{\partial F}{\partial y} = -\frac{4}{y^2} + \lambda = 0 \;\; ; \;\; \frac{\partial F}{\partial z} = -\frac{9}{z^2} + \lambda = 0. \tag{17}$$

Together with the constraint equation $x+y+z = 12$ this system can be easily solved by noting that the solutions should be positive. Therefore by eliminating $\lambda$ we find $x = y/2 = z/3$. The constraint equation is then $x + 2x + 3x = 6$, so $x = 2, y = 4, z = 6$ is a solution. Function takes minimum value of 3 here.

7. (Extra credit) Consider the vector $\vec{V} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$ and the scalar field $\psi(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$.

(a) show $\vec{\nabla}\times\vec{V} = -3\hat{k}$

$\epsilon_{ijk}\nabla_jv_k$ For $i = 1$, $\epsilon_{1jk}\nabla_jv_k = \nabla_2v_3 - \nabla_3v_2 = 0$; for $i = 2$, $\epsilon_{2jk}\nabla_jv_k = \nabla_3v_1 - \nabla_1v_3 = 0$; for $i = 3$, $\epsilon_{3jk}\nabla_jv_k = \nabla_1v_2 - \nabla_2v_1 = 1 - 4 = -3$. So answer is $-3\hat{z}$.

(b) evaluate $\int \vec{V}\cdot d\vec{r}$ from the origin (0,0,0) to (1,1,1) along the line $x = t$, $y = t^2$, $z = t^3$.

Since curl $\vec{v} \neq 0$, integral depends on path in general.

$$\begin{aligned}
\int\int\int_{0,0,0}^{1,1,1}(4y\hat{i} + x\hat{j} + 2z\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k}) &= \int_0^1(4t^2dt + t\cdot 2tdt + 2t^3\cdot 3t^2dt) \\
&= 6\left[\frac{1}{3}t^3 + \frac{1}{6}t^6\right]_0^1 = 3
\end{aligned} \tag{18}$$

(c) evaluate $\vec{\nabla}\psi$ and $\vec{\nabla} \times \vec{\nabla}\psi$.

$$\vec{\nabla}\psi = -(x, y, z)/(x^2 + y^2 + z^2)^{3/2} = -\frac{\vec{r}}{r^3} \tag{19}$$

and

$$\vec{\nabla} \times \vec{\nabla}\psi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x\psi & \partial_y\psi & \partial_z\psi \end{vmatrix}, \tag{20}$$

which vanishes by equality of mixed partial derivatives. You can also do this problem with $\epsilon_{ijk}$ notation if you like.

8. (Extra credit.) Calculate the radii of convergence of the following series:

(a)

$$\sum_{n=1}^{\infty} \frac{(nx)^n}{n!} \tag{21}$$

Use ratio test:

$$\begin{aligned} \rho_n &= \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}\frac{n!}{n^n x^n}\right| && (22) \\ &= \frac{|x|(n+1)^n}{n^n} = |x|\left(1 + \frac{1}{n}\right)^n && (23) \end{aligned}$$

$$\rho = \lim_{n\to\infty} \rho_n = |x| \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = |x|e \tag{24}$$

(b)

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2 + 1} \tag{25}$$

Use ratio test:

$$\begin{aligned} \rho_n &= \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{n^2 + 2n + 2}\frac{n^2 + 1}{x^2}\right| = |x|\frac{n^2 + 1}{n^2 + 2n + 2} && (26) \\ \rho &= \lim_{n\to\infty} \rho_n = |x| < 1 && (27) \end{aligned}$$