

## A Sample Calculation

This sample calculation contains just about everything that can come up in manipulating series: expanding functions (powers and logarithms) for small argument, multiplying two expansions, and substituting one series expansion into another series expansion. If you can do this one, you can do anything. The goal is to expand the function

$$f(x) = 1 + \frac{3}{x} + \frac{3(1+x)^{1/2}}{x^{3/2}} \ln \left[ (1+x)^{1/2} - x^{1/2} \right]$$

for small  $x$ . First, note that  $(1+x)^{1/2}$  appears more than once. From

$$(1+x)^p = 1 + px + \frac{1}{2}p(p-1)x^2 + \mathcal{O}(x^3),$$

we have

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^3 + \mathcal{O}(x^3).$$

Thus, the argument of the logarithm is  $1+y$ , where

$$y = -x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3).$$

From the series for the logarithm,

$$\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5 + \mathcal{O}(y^5),$$

and so in terms of  $x$  the logarithmic factor is

$$\begin{aligned} \ln(1+y) &= \left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right) - \frac{1}{2}\left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right)^2 \\ &\quad + \frac{1}{3}\left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right)^3 - \frac{1}{4}\left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right)^4 \\ &\quad + \frac{1}{5}\left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right)^5 + \mathcal{O}(x^3). \end{aligned}$$

We have already left out terms of order  $x^3$ , and so all the powers of  $y$  should be truncated at that order:

$$y^2 = x - x^{3/2} + \frac{1}{4}x^2 + \frac{1}{4}x^{5/2} + \mathcal{O}(x^3)$$

$$y^3 = -x^{3/2} + \frac{3}{2}x^2 - \frac{3}{4}x^{5/2} + \mathcal{O}(x^3)$$

$$y^4 = x^2 - 2x^{5/2} + \mathcal{O}(x^3)$$

$$y^5 = -x^{5/2} + \mathcal{O}(x^3)$$

and

$$\begin{aligned} \ln(1+y) &= \left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right) - \frac{1}{2}\left(x - x^{3/2} + \frac{1}{4}x^2 + \frac{1}{4}x^{5/2}\right) \\ &\quad + \frac{1}{3}\left(-x^{3/2} + \frac{3}{2}x^2 - \frac{3}{4}x^{5/2}\right)^3 - \frac{1}{4}\left(x^2 - 2x^{5/2}\right)^4 \\ &\quad + \frac{1}{5}\left(-x^{5/2}\right) + \mathcal{O}(x^3). \end{aligned}$$

In this expression, contributions linear in  $x$  appear in the first and second parentheses, with coefficients  $+\frac{1}{2}$  and  $-\frac{1}{2}$ , and so cancel. Contributions in  $x^2$  appear in each of the first four terms, with coefficients  $-\frac{1}{8}$ ,  $-\frac{1}{8}$ ,  $+\frac{1}{2}$ , and  $-\frac{1}{4}$ , and also cancel. This leaves  $\ln(1+y) = -x^{1/2} + \frac{1}{6}x^{3/2} - \frac{3}{40}x^{5/2} + \mathcal{O}(x^3)$ , and the full function is

$$\begin{aligned} f(x) &= 1 + \frac{3}{x} + \frac{3}{x^{3/2}} \left[ 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3) \right] \left[ -x^{1/2} + \frac{1}{6}x^{3/2} - \frac{3}{40}x^{5/2} + \mathcal{O}(x^3) \right] \\ &= 1 + \frac{3}{x} + \frac{3}{x} \left[ 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3) \right] \left[ -1 + \frac{1}{6}x - \frac{3}{40}x^2 + \mathcal{O}(x^3) \right] \\ &= 1 + \frac{3}{x} + \frac{3}{x} \left[ -1 - \frac{1}{3}x + \frac{2}{15}x^2 + \mathcal{O}(x^3) \right] = \frac{2}{5}x + \mathcal{O}(x^2). \end{aligned}$$

The terms of order  $1/x$  and the terms of order 1 cancel (as did several terms odd in  $x^{1/2}$ ).

This is not just an empty exercise; even if one were to evaluate the function numerically, two terms of order  $1/x$  must cancel to leave a result of order  $x$ , and with finite precision this eventually just doesn't work. For single precision floating point numbers, where the fractional accuracy of a given number is of order  $10^{-7}$ , the subtraction stops giving a meaningful answer when  $x = 10^{-3}$  as in the plot. The solid line shows the direct, single-precision evaluation of  $f(x)$ ; the dashed lines show the small- $x$  and large- $x$  approximations. The ‘‘approximation’’ is more accurate than the numerical evaluation for  $x \lesssim 10^{-3}$ .

