11/27/2017 Taylor: $$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n}$$

$$\left(a_n = \frac{d^n f}{dx^n}\bigg|_{x=0} \right)$$

Fourier: $$\boxed{f(x) = \sum_{m=-\infty}^{\infty} C_m \cdot e^{im(2\pi x/L)}}$$

$$C_m = \frac{1}{L}\int dx\, f(x)\, e^{-im(2\pi x/L)}$$

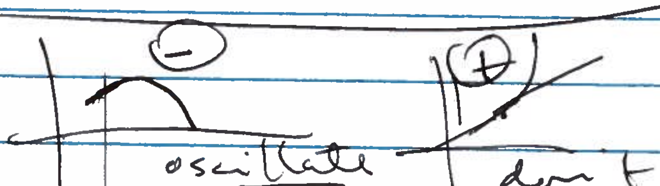
Two examples of general.

$$f(x) = \sum_n c_n \phi_n(x)$$

ϕ_n's often show up as solutions to second-order differential equations

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c\cdot y = (\text{something})$$

All 2nd order are harmonic oscillators, $y'' = \pm k^2 y$.



Second order diff. eq.

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y(x) = \lambda \cdot \rho(x) \cdot y(x).$$

"Sturm Liouville operator" (17.33).

$\mathcal{L}y = \lambda \rho y$ eigenvalue-problem.

$$\int dx \cdot (\mathcal{L}y)\, y' = \int dx\, \lambda \rho y y'$$

$$= \int dx \left[\frac{d}{dx}\left(p \frac{dy}{dx}\right) + q y\right] y'$$

$$= \cancel{\left[p \frac{dy}{dx} y'\right]}^{\,0} - \int dx \cdot p \frac{dy}{dx}\frac{dy'}{dx} + \int dx\, q y y'$$

$y \leftrightarrow y'$ $\quad -\int dx\, p \frac{dy'}{dx}\frac{dy}{dx} + \int dx\, q\, y' y = \int dx \cdot \lambda' \rho y' y$

subtract. $\quad \int dx\, (\lambda - \lambda')\rho y y' = 0$

$(\lambda \neq \lambda')$ $\quad \boxed{\int dx\, \rho y y' = 0}$ weighted inner product.

$y' = y^*$ $\quad \lambda = \lambda^*$ $\quad$ eigenvalues real.

$$\int dx\, (\lambda - \lambda^*)\rho |y|^2 = 0.$$

Bessel function

$$\frac{d^2 J_\nu}{dx^2} + \frac{1}{x}\frac{dJ_\nu}{dx} + \left(\alpha^2 - \frac{\nu^2}{x^2}\right) J_\nu = 0$$

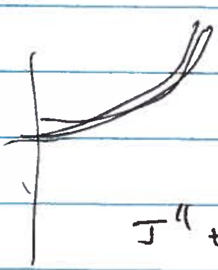
$$\frac{d}{dx}\left[p\frac{dJ}{dx}\right] = p\cdot\frac{d^2J}{dx^2} + \left(\frac{dp}{dx}\right)\frac{dJ}{dx}$$

$$= \frac{1}{x}\frac{d}{dx}\left(x\frac{dJ}{dx}\right)$$

$$\frac{d}{dx}\left(x\frac{dJ}{dx}\right) + \left(\alpha^2 x - \frac{\nu^2}{x}\right)J = 0.$$

$p(x) = x$ $\qquad$ $q(x) = -\frac{\nu^2}{x}$ $\qquad$ $p = 1.\ \lambda = \alpha^2$

$x < \nu$



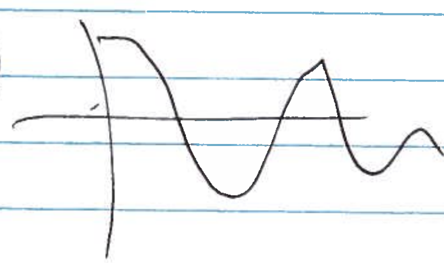
$J \sim x^{+\nu}$ $\qquad$ $x \to 0.$

$$J'' + \frac{J'}{x} - \frac{\nu^2 J}{x^2} = 0$$

$$p(p\cancel{-1})\cancel{+p} - \nu^2 = 0$$

$p^2 = \nu^2$

$x > \nu$



$$J'' + \frac{1}{x}J' + \alpha^2 J = 0$$

$$J \sim x^{-1/2}\begin{pmatrix}\sin \alpha x \\ \cos \alpha x\end{pmatrix}$$

Useful class polynomial solutions

$\{x^n\} \rightarrow$ power series, not eigenfunctions

Hermite polynomials. $H_n(x)$

Chebyshev polynomials $T_n(x)$

Legendre polynomials. $P_n(x)$

Laguerre polynomials $L_n(x)$.

$$\frac{d^2 H_n}{dx^2} - 2x\frac{dH_n}{dx} + \lambda H_n = 0 \qquad \lambda = 2n$$

$p = e^{-x^2}$ $\qquad \frac{d}{dx}\left(e^{-x^2}\frac{dH}{dx}\right) = e^{-x^2}\frac{d^2H}{dx^2} - 2xe^{-x^2}\frac{dH}{dx}$

$q = 0$ $\qquad \rho = e^{-x^2}$ $\qquad \boxed{\text{Table 17.1}}$

$-\infty < x < \infty$ $\qquad e^{-x^2}$ makes things converge.

normalization . $2^n x^n + \ldots$

③

generating function

$$\boxed{\exp\left(2xt - t^2\right) = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n!}}$$

$$\boxed{H_n(x) = \frac{d^n}{dx^n}\exp\left(2xt - t^2\right)\Big|_{t=0}}$$

$$H_n(x) = \frac{1}{2\pi i}\, n! \oint e^{2xt - t^2}\, t^{-(n+1)}\, dt$$

(wraps origin).

$H_0 = 1$. (always?)

$H_1 = 2x$

$H_2 = 4x^2 - 2$

$H_3 = 8x^3 - 12x$

odd / even.

$\int dx\, e^{-x^2} H_n H_m = (\ )\cdot \delta_{mn}$.

$$\boxed{H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right)}$$

coeff $x^n = 2^n$

$$\int dx\, e^{-x^2} H_n H_m = \delta_{mn}\, 2^n n! \sqrt{\pi}$$

6

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 = E_n\psi.$$

y (dimensionless) $= \sqrt{\frac{m\omega}{\hbar}}\cdot x = x/\alpha$ $\quad (x = \alpha y)$

$$\frac{\hbar^2}{2m}\frac{1}{\alpha^2} = \frac{1}{2}m\omega^2\alpha^2 \qquad \alpha^4 = \frac{\hbar^2}{m^2\omega^2} \qquad \alpha = \sqrt{\frac{\hbar}{m\omega}}.$$

$$\boxed{\psi_n = (\text{norm})\cdot e^{-\frac{1}{2}\frac{x^2}{\alpha^2}} H_n\left(\frac{x}{\alpha}\right)}$$

$$\cancel{-\frac{d^2\psi}{dy^2} + y^2\psi =}$$

$$-\frac{\hbar^2}{2m}\frac{1}{\alpha^2}\frac{d^2\psi}{dy^2} + \frac{1}{2}m\omega^2\cdot\alpha^2 y^2$$

$$= \frac{1}{2}\left(m\omega^2\cdot\frac{\hbar}{m\omega}\right)\left(-\frac{d^2\psi}{dy^2} + y^2\right) = E_n\psi$$

$$\boxed{\frac{d^2\psi}{dy^2} - y^2\psi = \frac{2E_n}{(\hbar\omega)}\psi}$$

$$\psi = e^{-y^2/2}H \qquad \psi' = \left((-y)H + H'\right)e^{-y^2/2}$$

$$\psi'' = (-H - yH' + H'')e^{-y^2/2}$$
$$-y(-yH + H')e^{-y^2/2}$$

$$\psi'' - y^2\psi = e^{-y^2/2}(H'' - 2yH' - H) = (2n)$$

$E_n = (n + \frac{1}{2})\hbar\omega.$

Start w/. $H_0 = 1$

$H_1 = a + bx$

$\int dx\, e^{-x^2} \cdot 1 \cdot (a + bx) = 0 = \sqrt{2\pi} \cdot a$ $\quad a = 0$

$b = 2$ — convention $\quad \boxed{H_1 = 2x}$

$\int dx\, e^{-x^2} \cdot 1 \cdot (a + \cancel{bx} + cx^2) = 0$

$\sqrt{\pi} \cdot a + \frac{1}{2} c \sqrt{\pi}$. $\quad c = -2a$

$\int dx \cdot e^{-x^2} \cdot 2x\, (\cancel{a} + bx + \cancel{cx^2}) = b \cdot \sqrt{\pi} = 0$. $\quad b = 0$

$\boxed{H_2 = 4x^2 - 2}$

H_{odd} = odd

H_{even} = even

(T) Chebyshev. Table 17.1

$p = \sqrt{1-x^2}$. $q = 0$ $\lambda = \nu^2$ $\rho = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\,\frac{dT}{dx}\right) + \nu^2 \cdot \frac{1}{\sqrt{1-x^2}}\, T_\nu = 0$$

$$\sqrt{1-x^2}\, T'' + \frac{1}{2}\frac{(-2x)}{\sqrt{1-x^2}}\, T' + \frac{\nu^2}{\sqrt{1-x^2}}\, T = 0$$

$$\boxed{(1-x^2)T'' - T' + n^2 T = 0} \qquad -1 < x < 1$$