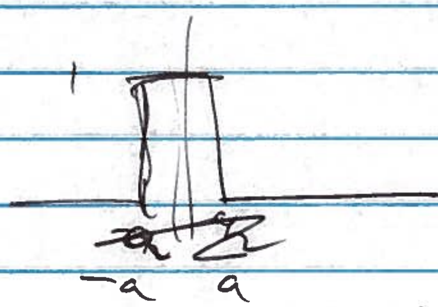
11/17/15 Fourier transform. Ch. 13.

$$\boxed{\begin{aligned} f(x) &= \int \frac{dk}{2\pi}\, \tilde{f}(k)\, e^{ikx} \\ \tilde{f}(k) &= \int dx\, f(x)\, e^{-ikx} \end{aligned}}$$

Lorentzian (Cauchy) $\quad f(x) = \dfrac{1}{x^2+a^2} \quad \longleftrightarrow \quad \tilde{f}(k) = \dfrac{\pi}{a}\, e^{-|k|a}$

Gaussian $\quad f(x) = e^{-\frac{1}{2}\frac{x^2}{a^2}} \quad \longleftrightarrow \quad \tilde{f}(k) = 2\pi \cdot \dfrac{a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}k^2a^2}$

Top-Hat



$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

$$\int dx\, f(x)\, e^{-ikx} = \left[\frac{e^{-ikx}}{-ik}\right]_{-a}^{a} = \frac{1}{-ik}\left(e^{-ika} - e^{+ika}\right)$$

$$= 2a \cdot \left(\frac{e^{ika} - e^{-ika}}{2ika}\right) = 2a \left(\frac{\sin ka}{ka}\right)$$

$a \to \infty \quad f(x) \to 1 \qquad \tilde{f} \to \infty \ (k=0). \qquad \int_{-\infty}^{\infty} dk\, 2a \frac{\sin ka}{ka} \qquad \boxed{\tilde{f} \to 2\pi\delta(k)}$

$\tilde{f} \to 0 \ (k \to \infty). \qquad = 2\int_{-\infty}^{\infty} dy \frac{\sin y}{y} = 2\pi$

other ~~limit~~. ~~exam~~:

mathworld. $\frac{\sin\left(\frac{x}{\epsilon}\right)}{\pi x} \to a\,\frac{\sin(ka)}{\pi ka} = \frac{1}{2\pi}\left[2a\,\frac{\sin ka}{ka}\right]$

$\epsilon = 1/a$

$$\boxed{\int \frac{dk}{2\pi}\, 1 \cdot e^{ikx} = \delta(x)}$$

δ is always a limit
background fn doesn't matter.

Iterate:

$$\tilde{f}(k) = \int dx'\, f(x')\, e^{-ikx'}$$

$$\int \frac{dk}{2\pi}\, \tilde{f}(k) = \int \frac{dk}{2\pi}\left[\int dx'\, f(x')\, e^{-ikx'}\right] e^{ikx}$$

$$= \int dx'\, f(x') \left[\int \frac{dk}{2\pi} \cdot e^{ik(x-x')}\right]$$

$$= \int dx'\, f(x')\, \delta(x-x') = \underline{f(x)}.$$

$(-i)$ on one, $(+i)$ on other $\to$ $\underline{\delta(x-x')}$.

reverse. $\to$ $\underline{\delta(k-k')}$

3

"convolution" [§13.1.7]

$$\int \frac{dk}{2\pi}\, \tilde{f}(k)\, \tilde{g}(k)\, e^{ikx}$$

$$= \int \frac{dk}{2\pi} \left[\int dx' f(x')\, e^{-ikx'}\right] \left[\int dx'' g(x'')\, e^{-ikx''}\right] e^{ikx}$$

$$= \int dx'\, dx''\, f(x')\, g(x'') \left[\int \frac{dk}{2\pi}\, e^{ik(x-x'-x'')}\right]$$

$$= \int dx'\, dx''\, f(x')\, g(x'')\,.\, \delta(x-x'-x'')$$

do x'' integral $\quad x'' = x - x'$

$$= \int dx'\, f(x')\, g(x-x')$$

do x' integral $\quad x' = x - x''$

$$= \int dx''\, f(x-x'')\, g(x'')$$

related by change of variable

"convolution"

g : p.s.f.

$f * g = f$ smeared/averaged over g.

(§13.1.7)

let $g = f^*(-x')$

$$\tilde{g} = \int dx' \, f^*(-x') \, e^{-ikx'}$$

$$= \int dx'' \, f^*(x'') \, e^{+ikx''} = \left[\int dx'' \, f(x'') \, e^{-ikx''} \right]^*$$

convolution:

$$\int dx' \, f(x') \, g(x-x') = \int dx' \, f(x') \, f^*(x'-x)$$

$$= \int \frac{dk}{2\pi} \tilde{f}(k) \tilde{g}(k) e^{ikx} = \int \frac{dk}{2\pi} \tilde{f}(k) \tilde{f}^*(k) e^{ikx}$$

let $x=0$

$$\int dx' \, f(x') f^*(x') = \int \frac{dk}{2\pi} \tilde{f}(k) \tilde{f}^*(k)$$

$$\boxed{\int dx \, |f(x)|^2 = \int \frac{dk}{2\pi} |\tilde{f}(k)|^2}$$

Parseval's Theorem. (13.46)

③

$$\int dx \left[\frac{1}{x^2+a^2}\right]^2 = \int \frac{dk}{2\pi} \cdot \left[\frac{\pi}{a} e^{-|k|a}\right]^2$$

$$= \frac{\pi^2}{2\pi a^2} \int_{-\infty}^{\infty} dk\, e^{-2|k|a} = \frac{\cancel{2}\pi^{\cancel{2}}}{\cancel{2}\cancel{\pi}a^2} \int_0^{\infty} dk \cdot e^{-2ka}$$

$$= \frac{\pi}{a^2} \left[\frac{e^{-2ka}}{-2a}\right]_0^{\infty} = \frac{\pi}{a^2}\left(\frac{0-1}{-2a}\right) = \frac{\pi}{2a^3}$$

$$\int \frac{dk}{2\pi} \left[2a \frac{\sin ka}{(ka)}\right]^2 = \frac{4a}{2\pi} \int dy \cdot \frac{\sin^2 y}{y^2}$$

$$= \int_{-a}^{a} dx\, (1)^2 = 2a.$$

$$\boxed{\int_{-\infty}^{\infty} dy \cdot \frac{\sin^2 y}{y^2} = \pi}$$

$$f(0) = \int \frac{dk}{2\pi} \left(2a \cdot \frac{\sin ka}{ka}\right) \cdot \cancel{e^{ikx}} = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \cdot \frac{\sin y}{y} = 1$$

$$\boxed{\int_{-\infty}^{\infty} dy \cdot \frac{\sin y}{y} = \pi}$$

Laplace Transform

$$\tilde{f}(k) = \int dx\, f(x)\, e^{-ikx} \qquad \left(\begin{array}{l} x \to t \\ k \to -is \end{array} \right)$$

$$\boxed{\tilde{f}(s) = \int_0^\infty dt\, f(t)\, e^{-st}}$$

$$f(x) = \int \frac{dk}{2\pi}\, \tilde{f}(k)\, e^{+ikx} \longrightarrow$$

$$\boxed{f(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{ds}{2\pi i}\, \tilde{f}(s)\, e^{+st}}$$

complex phase rotation

$$\cancel{-s = e^{z}} \qquad \cancel{ds = e^{z} dz} \qquad \cancel{e^{st} = t^{z-1}}$$

$$t' = \ln t \qquad z = -s$$

$$\boxed{\begin{array}{l} \phi(z) = \displaystyle\int_0^\infty dt\, t^{z-1} f(t) \\[2ex] f(t) = \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dz}{2\pi i}\, t^{-z}\, \phi(z) \end{array}}$$

$$\frac{d}{dx} f(x) = \int \frac{dk}{2\pi} \tilde{f}(k)\,(ik)\, e^{ikx}$$

$$\widetilde{\left[f'(x) \right]} = ik\tilde{f}$$

Sharpening

$$\cancel{f_{obs} = \int dx' \, f(x+x')\, W(x')}$$

$$f_{obs} = \int dx' \, f(x+x')\, W(x') \qquad \text{(symmetric)}$$

$$\Rightarrow \quad \tilde{f}_{obs} = \tilde{f}_{ideal}(k)\, \tilde{w}(k)$$

$$\Rightarrow \quad \tilde{f}_{ideal}(k) = \frac{\tilde{f}_{obs}(k)}{\tilde{w}(k)}.$$

Sharpening

gaussian PSF. → enhances high frequencies

Amplifies noise.