11/15/2017 $(0, 2\pi)$ $L = -i\partial_\phi$ $\rightarrow$ eigenvalues m

eigenvectors $e^{im\phi}$

stretch to $(0, L)$.

$$\boxed{f(x) = \sum_{m=-\infty}^{\infty} c_m \cdot e^{i\frac{2\pi x}{L}m}}$$

$$\int_0^L dx\, f(x)\, e^{-i\frac{2\pi}{L}px}$$

$$= \int_0^L dx \sum_{m=-\infty}^{\infty} c_m\, e^{i\frac{2\pi}{L}(m-p)x}$$

$$= \sum c_m \int_0^L dx\, e^{i\frac{2\pi}{L}(m-p)x} \quad m \neq p$$

$$= c_m \left[\frac{e^{i\frac{2\pi}{L}(m-p)x}}{i\frac{2\pi}{L}(m-p)}\right]_0^L$$

$$= \frac{c_m}{i\frac{2\pi}{L}(m-p)}\left[e^{i2\pi(m-p)} - 1\right] = 0$$

$$= \int_0^L dx \cdot c_p \cdot e^{i\frac{2\pi}{L}(p-p)x} = L \cdot c_p.$$

$$\boxed{f(x) = \sum_{m=-\infty}^{\infty} c_m \cdot e^{i\frac{2\pi m x}{L}}}$$

$$\boxed{c_m = \frac{1}{L} \cdot \int_{-L/2}^{L/2} dx\, f(x)\, e^{-i\frac{2\pi m x}{L}}}$$

$L \to \infty$ $\quad L c_m = \int_{-L/2}^{L/2} dx\, f(x)\, e^{-i k_m x}$ $\quad \boxed{k_m = \frac{2\pi}{L} \cdot m}$

$$\boxed{\tilde{c}(k_m) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-i k_m x}}$$

$$f(x) = \sum_{m=-\infty}^{\infty} c_m\, e^{i\left(\frac{2\pi m}{L}\right)x}$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \left(\frac{2\pi}{L}\right)(L c_m)\, e^{i\left(\frac{2\pi m}{L}\right)x}$$

$$= \frac{1}{2\pi} \sum_{m} \Delta k_m\, \tilde{c}(k_m)\, e^{i k_m x}$$

$$\longrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_m\, \tilde{c}(k_m)\, e^{i k_m x}$$

③

Rename

$$\boxed{f(x) = \int \frac{dk}{2\pi}\, \tilde{f}(k)\, e^{+ikx}}$$

$$\boxed{\tilde{f}(k) = \int dx\, f(x)\, e^{-ikx}}$$

Variations $x \leftrightarrow k$ variable names.

$i \leftrightarrow -i$ (complex conjugate)

(2π)'s move around

$\tilde{f} \to \cancel{\sqrt{2\pi}\,\tilde{f}}$ $\quad f(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{\tilde{f}(k)}{\sqrt{2\pi}}\, e^{+ikx}$

$$\frac{\tilde{f}(k)}{\sqrt{2\pi}} = \int \frac{dx}{\sqrt{2\pi}}\, f(x)\, e^{-ikx}$$

symmetric

$\frac{\tilde{f}}{\sqrt{2\pi}} \to \tilde{\tilde{f}}$

$$f(x) = \int \frac{dk}{\sqrt{2\pi}}\, \tilde{\tilde{f}}(k)\, e^{ikx} \qquad (13.5)$$

$$\tilde{\tilde{f}}(k) = \int \frac{dx}{\sqrt{2\pi}}\, f(x)\, e^{-ikx} \qquad (13.6)$$

wt

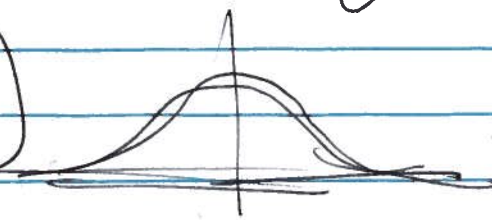
$k = 2\pi y$

$$\tilde{f}(y) = \int dx\, f(x)\, e^{-2\pi i x y}$$

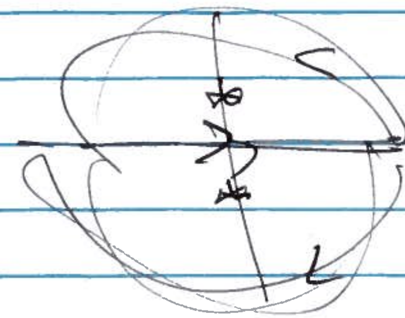
$$f(x) = \int dy\, \tilde{f}(y)\, e^{+2\pi i x y}$$

$k \to \omega$
$y \to f$

Example 1. $f(x) = \frac{1}{x^2+a^2}$



$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ikx}$$



poles $z = \pm ia$

closure depends on k.

$(k>0)$ $e^{ikx} = e^{ik(x+iy)} = e^{ikx}\, e^{-ky}$

need $y = \text{Im}\, z > 0$ exponentially damped.

$$\int \frac{dz}{z^2+a^2}\, e^{+ikX}\, e^{-kY} \to 0.$$

wraps pole @ $z = +ia$.

$$\oint \frac{dz\, e^{ikz}}{z^2+a^2} \to 2\pi i \cdot \frac{e^{ik(ia)}}{(2ia)} = \frac{\pi}{a} e^{-ka}$$

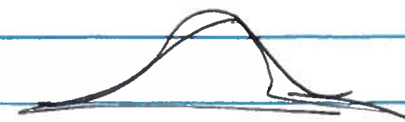
$(k<0)$ $(Y<0)$ pole $z = -ia$

$$\oint \frac{dz\, e^{ikz}}{z^2+a^2} \to (-)\, 2\pi i\, \frac{e^{ik(-ia)}}{(-2ia)} = \frac{\pi}{a} e^{+ka} = \frac{\pi}{a} e^{-|k|a}$$

clockwise

Gaussian

$f(x) = e^{-\frac{1}{2}x^2/a^2}$



$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2}\frac{x^2}{a^2}} \; e^{-ikx}$$

$$= \int_{-\infty}^{\infty} dx \; \exp\left\{-\frac{1}{2}\frac{1}{a^2}\left[x^2 + ikx\cdot 2a^2 + (ika)^2 - (ika)^2\right]\right\}$$

$$= \int_{-\infty}^{\infty} dx \; \exp\left[-\frac{1}{2a^2}(x+ika^2)^2 + \frac{1}{2a^2}\cdot(ika^2)^2\right]$$

$$= \exp\left(-\frac{1}{2}k^2a^2\right)\cdot \int_{-\infty}^{\infty} dx \; \exp\left(-\frac{1}{2a^2}(x+ika^2)^2\right)$$

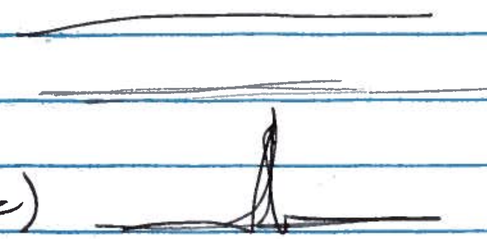
Cauchy's Theorem

$$\rightarrow \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2a^2}x^2} = \sqrt{2\pi}\cdot a.$$

$$\boxed{\tilde{f}(k) = \sqrt{2\pi}\, a \, e^{-\frac{1}{2}k^2a^2}}$$

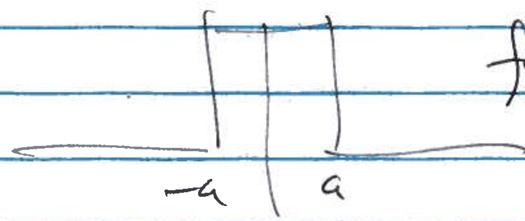
$$= (2\pi)\, \frac{a}{\sqrt{2\pi}}\, e^{-\frac{1}{2}k^2a^2}$$

$(a \to \infty)$. $f(x) \to 1$.



$\tilde{f}(k) \to 2\pi \cdot \delta(k)$

"Top hat".



$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a. \end{cases}$$

$$\int dx\, f(x)\, e^{-ikx} = \left[\frac{e^{-ikx}}{-ik} \right]_{-a}^{a}$$

$$= \frac{1}{-ik}\left(e^{-ika} - e^{ika} \right) = \frac{2a}{2ika}\left(e^{ika} - e^{-ika} \right)$$

$$\boxed{\tilde{f}(k) = 2a \cdot \left(\frac{\sin ka}{ka} \right)}$$

$(k \to 0)$ $\tilde{f} \to (2a)$. $\quad \mathrm{Im}(e^{iy})$

$$\int_{-\infty}^{\infty} 2a \cdot \frac{\sin ka}{(ka)}\, dk = 2 \int_{-\infty}^{\infty} \frac{\sin y}{y}\, dy.$$

$$= \mathrm{Im} \left[2 \cdot 2\pi i \left(\frac{1}{2} \right) \right] = 2\pi$$

$(a \to \infty)$ $\quad \tilde{f} \to (\infty)$ $(k = 0)$ $\quad \int dk \to 2\pi \quad \tilde{f} \to 2\pi\, \delta(k)$.

$\tilde{f} \to 0 \left(\frac{1}{ka} \right)$ $k \neq 0$