11/13/2017

Fourier series.

$0 < x < L$

$-L/2 < x < L/2$

(periodic)

Ch. 12.

$$f(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi r x}{L}\right) + b_r \sin\left(\frac{2\pi r x}{L}\right)\right]$$

(12.4)

formulas.

$$\int_0^L dx\, f(x) \cos\frac{2\pi p x}{L} = \int dx \left(\cdots + a_p \cos^2\frac{2\pi p x}{L} + \cdots\right)$$

$$= a_p \left(L \cdot \frac{1}{2}\right) \qquad \langle \cos^2 \rangle = \frac{1}{2}$$

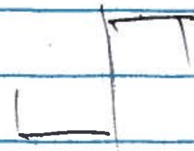
$$a_p = \frac{2}{L}\int_0^L dx\, f(x) \cos\frac{2\pi p x}{L}$$

(works for a_0).

$$b_p = \frac{2}{L}\int_0^L dx\, f(x) \sin\frac{2\pi p x}{L}$$

even function $\longrightarrow \{a_n\}$

odd function $\longrightarrow \{b_n\}$

square wave  odd $\quad a_n = 0$

$$\boxed{b_n = \frac{4}{\pi n}} \quad (n \text{ odd})$$

HW: continuous integrand

$$f(x) = \sum_{n \text{ odd}}^{\infty} \frac{4}{\pi}\frac{1}{n}\sin\left(\frac{2\pi n x}{L}\right)$$

$$\to \sum_n \frac{4}{\pi}\frac{1}{n}\sin\left(\frac{2\pi n t}{T}\right) = \sin(n\omega t)$$

integrate / differentiate term by term

$$\frac{4}{\pi n}\sin(nx) = \frac{d}{dx}\left(-\frac{4}{\pi n^2}\cos nx\right)$$

$$+1 = \frac{d}{dx}(x)$$
$$-1 = \frac{d}{dx}(-x)$$

square wave $= \frac{d}{dx}\left(|x|\right)$

$$\boxed{f(x) = \frac{|x|}{\pi} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos nx}$$

$$\int_{-\pi}^{\pi} \frac{|x|}{\pi}\cos nx\,dx = 2\int_0^{\pi} \frac{x}{\pi}\cos nx\,dx$$

$$= \frac{2}{\pi^2}\frac{1}{n^2}\Big(n\pi \underbrace{\cancel{\sin n\pi}}_{0} - \underbrace{\cos n\pi}_{(-1)^n} - 1\Big)$$

$f(0) = \ldots$ $\quad \frac{\pi^2}{8} = \sum_{n \text{ odd}} \frac{1}{n^2}$ $\quad \zeta(2) = \frac{\pi^2}{6}$ $\quad$ odd $= \left(1 - \frac{1}{2^2}\right)\zeta(2)$

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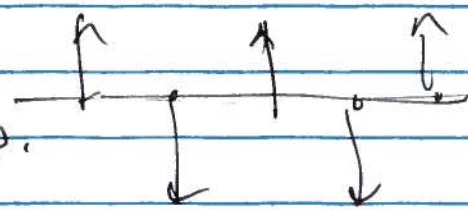
$\frac{d}{dx}$(square wave). $\boxed{f(x) = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4}{\pi} \cos nx}$

$x=0$ $\quad \sum_n \frac{4}{\pi}(1) \to \infty.$



$x = \pm\pi$ $\quad \sum_n \frac{4}{n}(\cos \cdot \text{odd}\,\pi) \to -\infty.$

$\frac{d}{dx}(\text{step}) = 2\delta(x).$

$$\delta(x) - \delta(x-\pi) = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\frac{b_p}{\pi} = \int_{x_0}^{x_0+2\pi} dx \cdot \left(2\delta(x) - 2\delta(\pi - x)\right) \sin px = \sin(p\cdot 0) - \sin(p\cdot\pi) = 0$$

$$= \cancel{\frac{b_n}{\pi}}$$

$$\frac{a_p}{\pi} = \int_{x_0}^{x_0+2\pi} dx \left(2\delta(x) - 2\delta(\pi - x)\right) \cos px = 2\left(\cos(p\cdot 0) - \cos p\cdot\pi\right)$$

$$= 2\left(1 - (-1)^p\right) = \begin{cases} 4 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

$$\boxed{\delta\text{-series} = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4}{\pi} \cos nx}$$

$\frac{4}{\pi} = \underline{1.27324}$

partial sum:

$$\sum_{k=0}^{M} \frac{4}{\pi} \cos\left((2k+1)x\right) = \mathrm{Re}\left[\frac{4}{\pi} \sum_{k=0}^{M} e^{i(2k+1)x}\right]$$

$$= \frac{4}{\pi} \cdot \mathrm{Re}\left[e^{ix} \sum_{k=0}^{M} \left(e^{2ix}\right)^{k}\right]$$

$$= \frac{4}{\pi} \mathrm{Re}\left(e^{ix} \cdot \frac{1-e^{(2ix)(M+1)}}{1-e^{2ix}}\right)$$

$$= \frac{4}{\pi} \mathrm{Re}\left[\frac{e^{2i(M+1)x} - 1}{e^{ix} - e^{-ix}}\right] = \frac{4}{\pi} \mathrm{Re}\left(\frac{e^{2i(M+1)x} - 1}{2i \sin x}\right)$$

$$= \frac{2}{\pi} \frac{\sin\left(2(M+1)x\right)}{\sin x}$$

$(x \to 0)$ $(x \to n\pi)$: $\frac{4}{\pi}(M+1)$ $(\pm)$

pointwise ~~converges~~ "converges in mean" (in integrals)

Parseval's Theorem : § 12.8

$$f^2(x) = \sum_n \left(a_n \cos nx + b_n \sin nx\right) \cdot \sum_m \left(a_m \cos mx + b_m \sin mx\right)$$

$$\int_{-L/2}^{L/2} dx \cdot f^2(x) = \int_{-L/2}^{L/2} dx \cdot \left[\frac{1}{2}a_0 + \sum_{m=1}^{\infty}\left(a_m \cos\frac{2\pi mx}{L} + b_m \sin\frac{2\pi mx}{L}\right)\right]$$
$$\times\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L}\right)\right]$$

$$= \int_{-L/2}^{L/2} dx \left[\left(\frac{1}{2}a_0\right)^2 + \sum_{m,n} a_m a_n \cos\frac{2\pi mx}{L}\cos\frac{2\pi nx}{L} + b_m b_n \sin\frac{2\pi mx}{L}\cdot\sin\frac{2\pi nx}{L}\right]$$

$$\langle \cos m \cdot \cos n \rangle = \frac{1}{2}\cdot\delta_{mn}$$

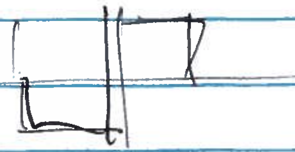
$$= \left(\frac{1}{2}a_0\right)^2 \cdot L + \sum_{n=1}^{\infty}\left(a_n^2\right)\left(\frac{L}{2}\right) + \left(b_n\right)^2\cdot\left(\frac{L}{2}\right)$$

$$= \cancel{\frac{L}{2}\left[\left(\frac{1}{2}a_0\right)^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]}$$

$$\boxed{\frac{1}{L}\int_0^L dx\,[f(x)]^2 = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

(12.13)

(6)

Square wave.  $f^2 = (\pm 1)^2 = 1.$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\,dx = 1 = \cancel{\left(\tfrac{1}{2}a_0\right)^2} + \sum_{n=1}^{\infty} \frac{1}{2}\left(\cancel{a_n^2} + b_n^2\right)$$

$$= \frac{1}{2}\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \left(\frac{4}{\pi n}\right)^2 = \frac{8}{\pi^2}\sum_{n\ \text{odd}} \frac{1}{n^2}$$

$$\boxed{\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}}$$ (again).

triangle $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{|x|}{\pi}\right)^2 dx = \frac{1}{2\pi}\cdot\frac{1}{\pi^3}\cdot\frac{2\pi^3}{3} = \frac{1}{3}$$

$$= \left(\frac{1}{2}\right)^2 + \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{1}{2}\left(\frac{4}{\pi^2 n^2}\right)^2 = \frac{1}{4} + \frac{8}{\pi^4}\sum_{\substack{n=1\\ \text{odd}}} \frac{1}{n^4}$$

$$\boxed{\sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{1}{n^4} = \left(\frac{1}{3} - \frac{1}{4}\right)\cdot\frac{\pi^4}{8} = \frac{\pi^4}{96}}$$

$$\left(1 - \frac{1}{2^4}\right)\left(\frac{\pi^4}{90}\right) = \frac{15}{16}\cdot\frac{\pi^4}{90}$$ ✓