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vectors

$$|\langle u|v\rangle|^2 \le \langle u|u\rangle \langle v|v\rangle$$ Schwartz remember.

(8.19) $\hat{a}, \hat{b}$

(17.12) f, g

operators

$$\langle(\Delta A)^2\rangle \langle(\Delta B)^2\rangle \ge \frac{1}{2}|\langle[A,B]\rangle|^2$$ (19.38)

$$\langle(\Delta p)^2\rangle \langle(\Delta x)^2\rangle \ge \left(\tfrac{1}{2}\hbar\right)^2$$

linear operators on vector spaces.

space → periodic functions $(0, 2\pi)$.

$$\boxed{L = -i\frac{\partial}{\partial\theta}} \qquad \langle\phi|\psi\rangle = \int_0^{2\pi} d\theta\, \phi^*(\theta)\, \psi(\theta)$$

$$\langle\phi|L\psi\rangle = \int_0^{2\pi} d\theta\, \phi^*(\theta) \left(-i\frac{\partial\psi}{\partial\theta}\right)$$

$$= \cancel{\left[-i\phi^*\psi\right]_0^{2\pi}} - \int_0^{2\pi} d\theta \left(-i\frac{\partial\phi^*}{\partial\theta}\right)\psi$$

= 0 periodic

$$= +\int_0^{2\pi} d\theta \left(-i\frac{\partial\phi}{\partial\theta}\right)^* \psi = \langle L\phi|\psi\rangle$$

$L^\dagger = L$ self adjoint.

Eigenvalues

$$L\psi = \lambda\psi = -i\frac{\partial\psi}{\partial\theta} \qquad \frac{\partial\psi}{\partial\theta} = i\lambda\psi$$

$$\boxed{\psi = e^{i\lambda\theta}}$$ not done yet.

ψ periodic $\Rightarrow$ λ real (self-adjoint)

period $= 2\pi$ $\Rightarrow$ λ = integer = m

$$\psi_m = C\, e^{im\theta}$$

Normalized: $\langle\psi_m|\psi_m\rangle = \int d\theta\, C^* e^{-im\phi}\, C e^{im\phi} = 2\pi|C|^2$

$$|C|^2 = \frac{1}{2\pi} \qquad C = \frac{1}{\sqrt{2\pi}}$$

$$\boxed{\psi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi}}$$

$$\langle\psi_m|\psi_n\rangle = \int d\theta \cdot \frac{1}{\sqrt{2\pi}} e^{-im\phi} \frac{1}{\sqrt{2\pi}} e^{+in\phi}$$

$$= \frac{1}{2\pi}\int d\theta\, e^{i(n-m)\theta} = \delta_{mn}$$

$$\{e^{im\theta}\} \longleftrightarrow \{\sin m\theta, \cos m\theta\}$$

$m = \dots, -2, -1, 0, 1, \dots$ $\qquad$ $m = (0), 1, 2, \dots$

particularly nice set of basis functions

$$\sum_{m=-\infty}^{\infty} c_m e^{im\theta} = c_0 + \sum_{m=1}^{\infty} \left(c_m e^{im\theta} + c_{-m} e^{-im\theta}\right)$$

$$= c_0 + \sum_{m=\infty}^{\infty} c_m(\cos m\theta + i \sin m\theta) + c_{-m}(\cos m\theta - i \sin m\theta)$$

$$= c_0 + \sum_{m=\infty}^{\infty} \overbrace{(c_m + c_{-m})}^{2\,\mathrm{Re}\, c_m} \cos m\theta + \underbrace{(c_m - c_{-m})}_{2\,\mathrm{Im}\, c_m} \sin m\theta$$

$\boxed{\text{Chapter 12}}$

$$\boxed{f(x) = \frac{1}{2} a_0 + \sum_{r=1}^{\infty} \left(a_r \cos r\theta + b_r \sin r\theta\right)}$$

(12.4) $\qquad$ $L = 2\pi$

else $\cos \frac{2\pi r x}{L}$

Some integrals :

$$\int_0^{2\pi} \sin nx \, dx = \left[-\frac{\cos nx}{n} \right]_0^{2\pi} = \frac{1 - \cos 2\pi n}{n} = 0$$

$$\int_0^{2\pi} \cos nx \cdot dx = \left[\frac{\sin nx}{n} \right]_0^{2\pi} = 0$$

$$\boxed{\langle \sin nx \rangle = \langle \cos nx \rangle = 0} \quad \begin{cases} \text{one period} \\ \text{several periods} \\ \text{Large interval} \quad \frac{1}{L} \int dx \end{cases}$$

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b.$$

$$\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$

$$\sin a \cdot \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$

$$\int_0^{2\pi} \cos mx \cdot \cos nx \cdot dx = \frac{1}{2} \int_0^{2\pi} \left[\cos(m-n)x + \cancel{\cos(m+n)x}^{\,0}\right] dx$$

$$= \begin{cases} 0 & m \neq n \\ \frac{1}{2} \cdot 2\pi & m = n \end{cases}$$

$$\int_0^{2\pi} \sin mx \cdot \sin nx \cdot dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} 2\pi & m = n. \end{cases}$$

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$m=n$. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\boxed{\langle \cos^2 x \rangle = \langle \sin^2 x \rangle = \frac{1}{2}}$$

$\sin a \cdot \cos b = \left(\sin(a+b) + \sin(a-b) \right)$

$$\langle \sin x \cos x \rangle = \langle \sin mx \cos nx \rangle = 0$$

given: $f(x)$

let $\bar{f}(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

let $\Theta^2 = \int_0^{2\pi} dx \left[f(x) - \bar{f}(x) \right]^2$

choose $\{a_n, b_n\}$ to minimize Θ^2.

$$\frac{\partial \Theta^2}{\partial a_0} = \int_0^{2\pi} dx \left[f - \bar{f}(x) \right] \left(\tfrac{1}{2}\right) \cdot 2$$

$$= \int_0^{2\pi} f(x)\,dx - \frac{1}{2} a_0 \cdot 2\pi$$

$$\boxed{\frac{1}{2} a_0 = \int_0^{2\pi} \frac{dx\, f(x)}{2\pi} = \langle f \rangle}$$

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$$\frac{\partial \theta^2}{\partial a_n} : \int_0^{2\pi} dx \cdot \left(f - \bar{f}\right) \cdot 2 \cdot \cos nx = 0$$

$$\int_0^{2\pi} dx\, f(x) \cos nx = \int_0^{2\pi} dx \left(\cancel{\tfrac{1}{2} a_0 + \cdots} + a_n \cos nx \cancel{+ \cdots} \right) \cos nx$$

$$= \frac{1}{2} \cdot a_n \cdot 2\pi = \pi a_n$$

$$\boxed{a_n = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x) \cdot \cos nx}$$

same formula works for a_0

$$\boxed{b_n = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x) \sin nx}$$

$$f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, f(x) \, dx = 0.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, \underbrace{f(x)}_{\text{odd}} \, \underbrace{\cos nx}_{\text{even}} = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \, f(x) \sin nx$$

$$= \frac{2}{\pi} \int_0^{\pi} dx \cdot \sin nx \, dx = \frac{2}{\pi} \left(\frac{-\cos nx}{n} \right)_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{-\cos n\pi + 1}{n} \right] = \frac{2}{\pi n} \left(1 - (-1)^n \right)$$

$$= \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$