11/3/2017 linear algebra.

$$\vec{u}\cdot\vec{v} = u_k^* v_k = u^\dagger v = (u,v) = \langle u,v\rangle$$

$$(u, Mv) = (u_i)^* M_{ij} v_j = (M_{ji}^* u_i^*) v_j$$

$$= (M^\dagger u, v)$$ adjoint operator

self adjoint: $(u, Mv) = (Mu, v)$

$$\langle u | Mv\rangle = \langle Mu | v\rangle = \langle u | M | v\rangle$$

QM: $\langle\psi|\psi\rangle$ $\langle\psi|M|\psi\rangle$
"Dirac Bracket"

functions $\psi(x), \phi(x) \leftrightarrow$ "vector space"

$$\langle\phi|\psi\rangle = \sum_k \phi_k^* \psi_k \rightarrow \sum_x \phi^*(x)\psi(x) = \int dx\, \phi^*(x)\psi(x)$$

$$\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$$

$$\langle\chi | \alpha\phi + \beta\psi\rangle = \alpha\langle\chi|\phi\rangle + \beta\langle\chi|\psi\rangle$$

$$\langle\alpha\chi|\phi\rangle = \alpha^*\langle\chi|\phi\rangle$$

As before: $M\phi = \lambda\phi$

$M\phi' = \lambda'\phi'$

$$\langle\phi'|M\phi\rangle = \lambda\langle\phi'|\phi\rangle$$

$$\langle\phi|M\phi'\rangle = \lambda'\langle\phi|\phi'\rangle = \langle M\phi|\phi'\rangle$$

$$\langle\phi'|M\phi\rangle^* = \lambda'\langle\phi'|\phi\rangle^*$$

$$\langle\phi'|M\phi\rangle = \lambda'^*\langle\phi'|\phi\rangle$$

$$\boxed{(\lambda - \lambda'^*)\langle\phi'|\phi\rangle = 0}$$

$\phi' = \phi$ λ real $\lambda = \lambda^*$

$\lambda \neq \lambda'$ $\langle\phi'|\phi\rangle = 0$

Schwarz inequality

$$\langle\phi - \alpha\psi|\phi - \alpha\psi\rangle = \int dx\,|\phi - \alpha\psi|^2 \geq 0. \quad \text{any } \alpha$$

$$\langle\phi|\phi\rangle - \alpha^*\langle\psi|\phi\rangle - \alpha\langle\phi|\psi\rangle + |\alpha|^2\langle\psi|\psi\rangle \geq 0$$

$\frac{\partial}{\partial\alpha^*} = 0$ $\langle\psi|\phi\rangle = \alpha\langle\psi|\psi\rangle$ $\quad \alpha = \frac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$

$\frac{\partial}{\partial\alpha} = 0$ $\langle\phi|\psi\rangle = \alpha^*\langle\psi|\psi\rangle$ $\quad \alpha^* = \frac{\langle\phi|\psi\rangle}{\langle\psi|\psi\rangle}$

(doesn't depend on how α obtained.)

$$\langle\psi|\psi\rangle - \left(\frac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}\right)\langle\psi|\phi\rangle$$

$$- \left(\frac{\langle\psi|\phi\rangle}{\langle\phi|\phi\rangle}\right)\langle\phi|\psi\rangle \geq 0$$

$$+ \frac{|\langle\phi|\psi\rangle|^2}{|\langle\phi|\phi\rangle|^2}\langle\phi|\phi\rangle$$

$$\langle\psi|\psi\rangle\langle\phi|\phi\rangle - |\langle\phi|\psi\rangle|^2 - \cancel{|\langle\psi|\phi\rangle|^2} + \cancel{\langle\psi|\phi\rangle^2}$$

$$= \langle\psi|\psi\rangle\langle\phi|\phi\rangle - |\langle\phi|\psi\rangle|^2 \geq 0$$

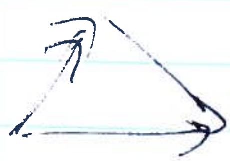
$$\boxed{|\langle\psi|\phi\rangle|^2 \leq \langle\psi|\psi\rangle\langle\phi|\phi\rangle}$$

§ 8.1.3

(8.1[illegible])

$$(\vec{A}\cdot\vec{B})^2 \leq \|\vec{A}\|^2\|\vec{B}\|^2$$

$$\cos^2\theta \leq 1$$



$$\|\vec{a}+\vec{b}\|^2 = \|a^2\| + \|b^2\| + 2(\vec{a}\cdot\vec{b})^2$$

$$\leq \|a\|^2 + \|b\|^2 + 2\|a\|^2\|b\|^2$$

triangle inequality

$$= (\|\vec{a}\| + \|\vec{b}\|)^2$$

(8.20)

(4)

$$\left[\int dx\, f(x)\, g(x)\right]^2 \le \int dx\, f^2(x) \int dx\, g^2(x)$$

$$f = x\sqrt{p(x)} \qquad g = \sqrt{p(x)}$$

$$\left(\int dx.\, x\, p(x)\right)^2 \le \int dx\, x^2 p(x) \int dx\, p(x)$$

$$\left[\frac{\int dx\, x\, p(x)}{\int dx\, p(x)}\right]^2 \le \int \frac{dx\, x^2 p(x)}{\int dx\, p(x)}$$

$$\boxed{\langle x^2 \rangle \ge \langle x \rangle^2}$$

<u>operators</u> $\hat{x} = x.$

$$\int dx\, \psi^*(x).\, x\,\psi(x) = \int dx\, x\,\psi^*(x).\,\psi(x)$$

$$\langle \psi, \hat{x}\psi \rangle = \langle \hat{x}\psi, \psi \rangle$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

$$\langle \psi | \hat{p}\psi \rangle \quad \int dx\, \psi^*(x)\left(-i\hbar \frac{\partial\psi}{\partial x}\right) = \left[-i\hbar\, \psi^*(x)\psi(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, (-i\hbar)\frac{\partial\psi^*}{\partial x}\psi$$

$$= +\int_{-\infty}^{\infty} dx \left(-i\hbar \frac{\partial\psi}{\partial x}\right)^* \psi = \langle \hat{p}\psi | \psi \rangle$$

(8)

$$\langle x \rangle = \langle \psi | x \psi \rangle = \int dx\, \psi^* x \psi$$

$$\langle (x - \langle x \rangle)^2 \rangle = \int dx\, (x - \langle x \rangle)^2 \, |\psi|^2$$

$$= \int dx \left(x^2 - 2x\langle x \rangle + \langle x \rangle^2 \right) |\psi|^2$$

$$= \int dx\, x^2 |\psi|^2 - 2\langle x \rangle \int dx\, x\, |\psi|^2 + \langle x \rangle^2 \int dx\, |\psi|^2$$

$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2 \geq 0$$

§ 13.1.1

$$\int dx\, |f|^2 \int dx\, |g|^2 \geq \left| \int dx\, f^* g \right|^2$$

$$f = (x - \langle x \rangle)\psi \qquad g = (p - \langle p \rangle)\psi$$

$$\int dx\, (x - \langle x \rangle)^2 |\psi|^2 \cdot \int dx\, (p - \langle p \rangle)^2 |\psi|^2$$

$$\geq \left| \int dx\, (x - \langle x \rangle)\psi^* \, (p - \langle p \rangle)\psi \right|^2$$

LHS $\rightarrow$ $\langle (\Delta x)^2 \rangle \ \langle (\Delta p)^2 \rangle$

RHS $\rightarrow$ $\left| \int dx\, \psi^* \, (x - \langle x \rangle)(p - \langle p \rangle)\psi \right|^2$

$$AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA).$$

$$(x - \langle x \rangle)(p - \langle p \rangle) = \frac{1}{2}\left(\begin{matrix}\text{symmetric}\\ \text{sum}\end{matrix}\right) + \frac{1}{2}[x, p]$$

$$(x - \langle x \rangle)(p - \langle p \rangle) - (p - \langle p \rangle)(x - \langle x \rangle)$$

$$= xp - \langle x \rangle p - x\langle p \rangle + \cancel{\langle x \rangle \langle p \rangle}$$
$$- px + p\langle x \rangle + \langle p \rangle x - \cancel{\langle p \rangle \langle x \rangle}$$

$$= \underline{xp - px}$$

$$(xp - px)\psi = \left(x\left(-i\hbar\frac{\partial}{\partial x}\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x\right)\psi$$

$$= -i\hbar\left(x\frac{\partial \psi}{\partial x}\right) + i\hbar\frac{\partial}{\partial x}(x\psi) \qquad = i\hbar\psi$$
$$\hookrightarrow \cancel{x\frac{\partial \psi}{\partial x}} + \psi$$

$$[x, p] = i\hbar$$

$$\underline{\text{LHS}} = \left|\frac{1}{2}\int dx\, \psi^* (F + i\hbar)\psi\right|^2$$

$$F = xp + px \qquad F^\dagger = x^\dagger p^\dagger + p^\dagger x^\dagger \qquad F^\dagger = F$$

$\underline{F^\dagger = F}$ $\rightarrow$ real eigenvalues $\rightarrow$ $\langle \psi | F \psi \rangle$ $\underline{\text{real}}$

$i\hbar \langle \psi | \psi \rangle$ $\underline{\text{imaginary}}$

$$\langle (\Delta x)^2 \rangle \langle (\Delta p)^2 \rangle \geq \frac{1}{4} \langle \hat{F} \rangle^2 + \frac{1}{4} \hbar^2 \geq \frac{1}{4} \hbar^2$$

$$\boxed{(\Delta x)(\Delta p) \geq \frac{1}{2} \hbar}$$

Heisenberg.