1/1/2007

Last time: "self-adjoint" $M^\dagger = M$

has real eigenvalues, orthogonal eigenvectors.

$M^\dagger = M$: $M_{ij}^* = M_{ji}$ real - symmetric

complex - "Hermitian"

Rotation:
$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z.$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$Rv = \lambda v$ $\quad \det(R - \lambda \mathbb{1}) = 0.$

$$\det \begin{pmatrix} \cos\theta-\lambda & \sin\theta & 0 \\ -\sin\theta & \cos\theta-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$= \left[(\lambda-\cos\theta)^2 + \sin^2\theta\right](1-\lambda)$$

$$\lambda^2 - 2\cos\theta + \cos^2\theta + \sin^2\theta = \lambda^2 - 2\cos\theta + 1$$

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4\cdot 1}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

mean = 60.8
median = (62)
std. dev. = 22.4
$z = -0.677$
$u = -0.105$

$$\boxed{e^{ix} = \cos x + i\sin x}$$

$F = \Phi + i\Psi$. $\vec{v} = \vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}$

$\vec{\nabla}\cdot\vec{v} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$.

$\vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\nabla}\Phi) = 0$.

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{1+x^2} \to \mathrm{Re} \oint \frac{e^{iz}\, dz}{1+z^2}$$

$e^{i(R\cos\theta + iR\sin\theta)}$

$= e^{iR\cos\theta}\, e^{-R\sin\theta}$

$\sin\theta > 0$

$$2\pi i \cdot \left[\cancel{(z-i)} \frac{e^{iz}}{(z+i)\cancel{(z-i)}} \right]_{z \to i} = 2\pi i \cdot \frac{e^{i(i)}}{2i} = \pi \cdot e^{-1} = \frac{\pi}{e}$$

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roots: $\lambda = 1$ $\lambda = e^{\pm i\theta}$

eigenvectors. $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ $\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$

axis

Any rotation leaves rotation axis fixed

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

projects $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$ projects $\perp$ z $P_{\perp}$

$\begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$ projects onto z $P_{\parallel}$.

$P^2 = P$ $\begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

$\lambda^2 = \lambda$ $\lambda = 0, \lambda = 1$ keep or annihilate

"idempotent"

Hermitian matrix:
can choose basis of eigenvectors.

$\hat{e}_1 = \vec{v}_1$ $\qquad$ $\hat{e}_2 = \vec{v}_2$ $\qquad$ $\hat{e}_3 = \vec{v}_3$ $\qquad$ (normalized somehow)

$$\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

change of basis = rotation

$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = R \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

$$R(\lambda_1 \vec{v}_1) = R \cdot M \vec{v}_1 = RM(R^\dagger R)\vec{v}_1 = (RMR^\dagger)(R\vec{v}_1)$$

$$\lambda (R\vec{v}_1) = (RMR^\dagger)(R\vec{v}_1)$$

(rotated matrix) acting on (rotated vector)
= (eigenvalue)(rotated vector)

real $\rightarrow$ orthogonal $\rightarrow$ $RR^T = 1$. $\quad R^T = R^{-1}$

complex $\rightarrow$ "unitary" $\quad UU^\dagger = 1 \quad U^\dagger = U^{-1}$

matrix = (transform) of $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$u^\dagger u = 1 \to \det u^\dagger \cdot \det u = (\det u)^* (\det u) = 1.$

$$|\det u|^2 = 1 \quad \checkmark \; \det u = e^{i\theta}$$

$\det (u m u^\dagger) = \det u \cdot \det m \cdot \det u^\dagger = \det m$

$$\checkmark \; \det m = \lambda_1 \cdots \lambda_N = \prod_{i=1}^{N} \lambda_N$$

$\mathrm{Tr.}\,(u m u^\dagger) = u_{ij} m_{jk} u^\dagger_{ki}$

$= u^\dagger_{ki} u_{ij} m_{jk} = \mathrm{tr}(u^\dagger u m) = \mathrm{Tr}\, m.$

$$\checkmark \; \mathrm{Tr.}\, m = \lambda_1 + \cdots + \lambda_N = \sum_{i=1}^{N} \lambda_i$$

$$\det (e^{m}) = (e^{\lambda_1} \cdots e^{\lambda_N}) = e^{(\lambda_1 + \cdots + \lambda_N)} = e^{\mathrm{Tr}\, m}$$

$R = e^{\theta J} \qquad \det R = 1 = e^{\theta \,\mathrm{tr}\, J} \qquad \mathrm{Tr}\, J = 0$

$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J^3 = -J \quad J^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\exp(\theta J) = \mathbb{1} + \theta J + \frac{1}{2}\theta^2 J^2 + \frac{1}{6}\theta^3 J^3 + \cdots$$

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad J^2 = \left(\begin{array}{cc|c} -1 & & \\ & -1 & \\ \hline & & \end{array}\right)$$

$$J^3 = \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & \\ \hline & & \end{array}\right) = -J \qquad J^4 = \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & \end{array}\right)$$

$$\exp(\theta J) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \theta \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix} + \frac{1}{2}\theta^2 \begin{pmatrix} -1 & & \\ & -1 & \\ & & \end{pmatrix}$$

$$+ \frac{1}{6}\theta^3 \left(\begin{array}{cc|c} & -1 & \\ 1 & & \\ \hline & & \end{array}\right) + \frac{1}{24}\theta^4 \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & \end{array}\right)$$

$$= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & \end{array}\right)\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \cdots\right)$$

$$+ \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & \end{array}\right)\left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots\right)$$

$$= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \left(\begin{array}{cc|c} & 1 & \\ -1 & & \\ \hline & & \end{array}\right)\sin\theta + \left(\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & \end{array}\right)\cos\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \; !$$