

1/1/2007

Last time: "self-adjoint" $M^t = M$

has real eigenvalues, orthogonal eigenvectors.

$M^t = M$: $M_{ij} = M_{ji}$ real-symmetric
 complex - "Hermitian"

Rotation: $x' = x \cos \theta + y \sin \theta$
 $y' = -x \sin \theta + y \cos \theta$
 $z' = z$.

$$R = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$Rv = \lambda v$ $\det(R - \lambda I) = 0$.

$$\det \begin{pmatrix} \cos \theta - \lambda & \sin \theta & 0 \\ -\sin \theta & \cos \theta - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix}$$

$$= [(\cos \theta - \lambda)^2 + \sin^2 \theta] (1 - \lambda)$$

$$\lambda^2 - 2 \cos \theta \lambda + \cos^2 \theta + \sin^2 \theta = \lambda^2 - 2 \cos \theta \lambda + 1$$

$$\lambda = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2}$$

$$= \cos \theta \pm i \sin \theta$$

$$\text{mean} = 60.8$$

$$\text{median} = (62)$$

$$\text{std. dev.} = 22.4$$

$$z = -0.677$$

$$u = -0.105$$

$$e^{ix} = \cos x + i \sin x$$

$$F = \Phi + i\Psi, \quad \vec{v} = \vec{\nabla}\Phi = \frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}$$

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\nabla}\Phi) = 0$$

$$\int_0^\infty \frac{\cos x}{1+x^2} dx \rightarrow \text{Re} \int \frac{e^{iz}}{1+z^2} dz$$

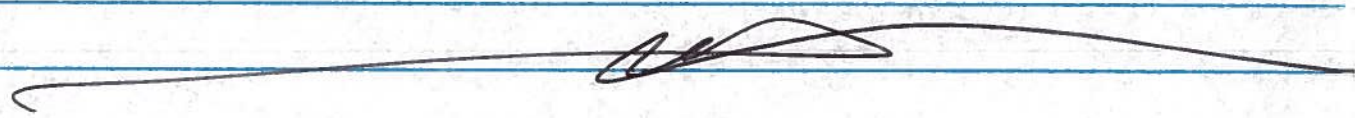
$$\frac{1}{2} (e^{i\pi/2} + e^{-i\pi/2}) \\ = e^{i\pi/2} - e^{-i\pi/2} \\ \text{for } \text{Im} z > 0$$

$$2\pi i \cdot \int_{\gamma} \frac{e^{iz}}{(z+i)(z-i)} dz = 2\pi i \cdot \frac{e^{-1}}{2i} = \pi \cdot e^{-1} = \frac{\pi}{e}$$

roots: $(\lambda = -1)$ $(\lambda = e^{\pm i\phi})$

eigenvectors: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$
ax=3

Any rotation leaves rotation axis fixed



$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

projects $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projects ⊥ z \mathbb{P}_1

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projects onto z \mathbb{P}_4

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

$\lambda = 0$

$\lambda = 0, \lambda = 1$

keep or annihilate

"idempotent"

(4)

Hermitian matrix:

can choose basis of eigenvectors.

$$\hat{e}_1 = \vec{v}_1$$

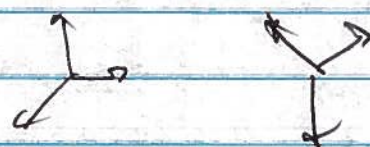
$$\hat{e}_2 = \vec{v}_2$$

$$\hat{e}_3 = \vec{v}_3$$

(normalized
orthonormal)

$$\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$$

change of basis = rotation



$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = R \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \quad \cancel{\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}} = R \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix}$$

$$R(\lambda \vec{v}_i) = R \cdot M \vec{v}_i = R M (R^T)^{-1} \vec{v}_i = (R M R^T) (R \vec{v}_i)$$

$$\lambda (R \vec{v}_i) = (R M R^T) (R \vec{v}_i)$$

(rotated matrix) acting on (rotated vector)

= (eigenvalue) (rotated vector)

real \rightarrow orthogonal $\rightarrow R R^T = I$ $R^T = R^{-1}$

complex \rightarrow "unitary" $U U^\dagger = I$ $U^\dagger = U^{-1}$

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matrix = (transform) of $(\lambda_1 \dots \lambda_n)$

$$U^T U = I \rightarrow \det U^T \cdot \det U = (\det U)^T (\det U) = 1.$$

$$|\det U|^2 = 1 \quad \sqrt{\det U = e^{i\theta}}$$

$$\det (U M U^T) = \det U \cdot \det M \cdot \det U^T = \det M$$

$$\sqrt{\det M = \lambda_1 \dots \lambda_n = \prod_{i=1}^n \lambda_i}$$

$$\text{Tr. } (U M U^T) = U_{ij} M_{jk} U_{ki}^T$$

$$= U_{ki}^T U_{ij} M_{jk} = \text{tr}(U^T U M) = \text{tr } M.$$

$$\text{tr } M = \lambda_1 + \dots + \lambda_n = \sum_{i=1}^n \lambda_i$$

$$\det (e^M) = (e^{\lambda_1} \dots e^{\lambda_n}) = e^{(\lambda_1 + \dots + \lambda_n)} = e^{\text{tr } M}$$

$$R = e^{\theta J}$$

$$\det R = 1 = e^{\theta \text{tr } J}$$

$$\text{tr } J = 0$$

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J^3 = -J$$

$$J^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

②

$$\exp(\theta J) = 1 + \theta J + \frac{1}{2} \theta^2 J^2 + \frac{1}{6} \theta^3 J^3 + \dots$$

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$J^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -J \quad J^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\exp(\theta J) = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} + \theta \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \theta^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$+ \frac{1}{6} \theta^3 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{24} \theta^4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \left(\theta - \frac{1}{6} \theta^3 + \frac{1}{120} \theta^5 + \dots \right)$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{2} \theta^2 + \frac{1}{24} \theta^4 + \dots \right)$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cos \theta + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos \theta$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}$$