10/20/2017 Cauchy integral formula.

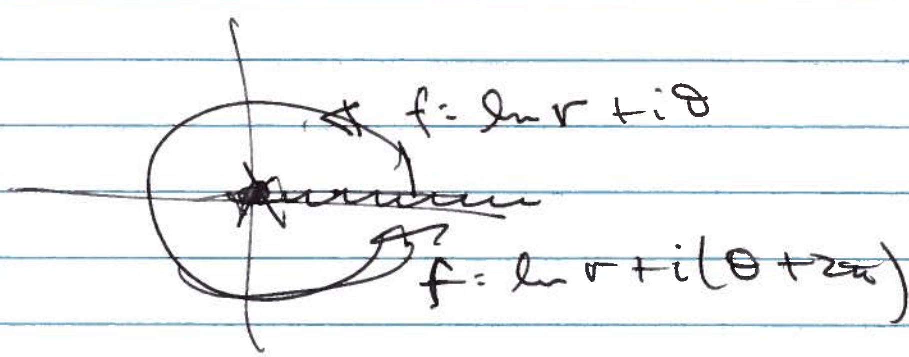
$$\oint_C dz\, f(z) = 2\pi i \sum_n R_n$$

$$\boxed{R_n = \lim_{z \to z_n} (z - z_n) f(z)}$$

Isolated singularities.

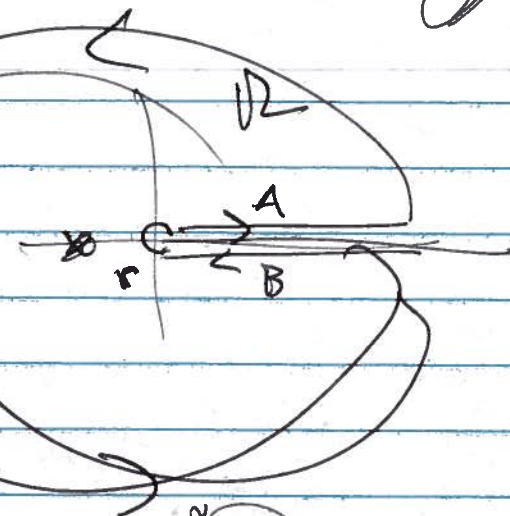
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2} = \oint \frac{dz}{(z-i)(z+i)} = 2\pi i \cdot \left(\frac{1}{2i}\right) = \pi$$

Branch cuts: $\ln z$, z^α



2

$$\int_0^\infty \frac{x^{\alpha-1}\,dx}{1+x}$$



$0<\alpha<1$:

$x\to 0$ $\int x^{\alpha-1}\,dx$ converges as $x^{\alpha}\to 0$

$x\to\infty$ $\int \frac{x^{\alpha-1}\,dx}{x} \to x^{\alpha-1} \to (x\to\infty)$.

pole: $z=-1 = e^{i\pi}$.

$$R(\pi) = (e^{i\pi})^{\alpha-1} = e^{i(\alpha-1)\pi} = \underline{\underline{-e^{i\alpha\pi}}}$$

$$\int_{\text{circle}} dz\, f(z) = \int_0^{2\pi} \frac{ir e^{i\theta}\,d\theta \cdot r^{\alpha-1} e^{i(\alpha-1)\theta}}{1+re^{i\theta}}$$

$$\to r^{\alpha}\int_0^{2\pi} i e^{i\alpha\theta}\,d\theta \qquad r\to 0$$

$$\to R^{\alpha-1}\int_0^{2\pi} i e^{i(\alpha-1)\theta}\,d\theta \qquad R\to\infty.$$

↑ something finite.

(B)

(A): $z = x = r \qquad \int_0^\infty \frac{r^{\alpha-1}\,dr}{1+r}. \qquad \xrightarrow{\text{(A)}}$

(B) $z = r e^{i2\pi} \qquad - \int_0^\infty \frac{r^{\alpha-1} e^{i2\pi(\alpha-1)}}{1 + r e^{i2\pi}}\,dr \qquad \xleftarrow{\text{(B)}}$

$-1.$

$$= - \int \frac{r^{\alpha-1}\,dr}{1+r}. \times e^{2\pi i \alpha}$$

$$\oint f(z)\,dz = 2\pi i.\left(-e^{i\alpha\pi}\right)$$

$$= (A) + (B) + \cancel{(r)} + \cancel{(R)}$$

$$= \left(1 - e^{2\pi i\alpha}\right) \int_0^\infty \frac{r^{\alpha-1}\,dr}{1+r}$$

$$\int_0^\infty \frac{x^{\alpha-1}\,dx}{1+x} = \frac{(2\pi i)\left(-e^{i\alpha\pi}\right)}{\left(1 - e^{2\pi i\alpha}\right)}$$

$$= \left(\frac{2i}{e^{i\pi\alpha} - e^{-i\pi\alpha}}\right). \pi = \frac{\pi}{\sin\alpha\pi}$$

$$\boxed{\int_0^\infty \frac{x^{\alpha-1}\,dx}{1+x} = \frac{\pi}{\sin(\alpha\pi)}}$$

$\int_0^\infty x^{\alpha-1} \sin x \, dx$ — Branch cut (don't wrap origin) — finite.

$C =$ 

Im. $z^{\alpha-1} e^{iz}$

$R \to \infty$

$$\oint_C z^{\alpha-1} e^{iz} dz = \int_0^\infty x^{\alpha-1} \sin x \, dx + \int_0^{\pi/2} iRe^{i\theta} d\theta \, (Re^{i\theta})^{\alpha-1} \exp(iRe^{i\theta})$$

$e^{-R\sin\theta}$

$$+ \int_\infty^0 (i\,dy)(iy)^{\alpha-1} e^{i(iy)}$$

$\to 0$

$$\int_0^\infty x^{\alpha-1} \sin x \, dx = \int_0^\infty i\,dy \cdot i^{\alpha-1} y^{\alpha-1} e^{-y}$$

$$\text{Im} = i\left(e^{i\frac{\pi}{2}}\right)^{\alpha-1} \underbrace{\int_0^\infty y^{\alpha-1} dy \, e^{-y}}_{\Gamma(\alpha)}$$

$$= i\left(\cos\left((\alpha-1)\frac{\pi}{2}\right) + i \sin\left((\alpha-1)\frac{\pi}{2}\right)\right)$$

$$\boxed{\int_0^\infty x^{\alpha-1} \sin x \, dx = \cos\left(\frac{\alpha\pi}{2} - \frac{\pi}{2}\right) \Gamma(\alpha)}$$

$$\cos\frac{\alpha\pi}{2} \cos\frac{\pi}{2} + \sin\frac{\alpha\pi}{2} \cdot \sin\frac{\pi}{2}$$

$$\boxed{\sin\frac{\alpha\pi}{2} \cdot \Gamma(\alpha)} \qquad \boxed{-1 < \alpha < 1}$$

mathematica