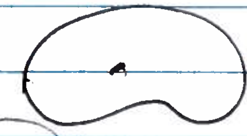
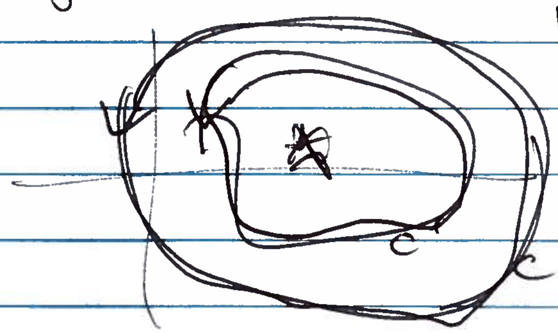
10/18/2017 Analytic function of complex variable.

$$\boxed{\oint_C f(z)\,dz = 0}$$



(24.40)

Singularities can give non-zero result



$$\boxed{\oint_C f(z)\,dz = \oint_{C'} f(z)\,dz}$$

~~if deformation~~ (doesn't cross singularity)

(24.45)

$$\oint_C \frac{f(z)\,dz}{(z-z_0)} = 2\pi i\, f(z_0)$$

(C wraps z_0).

(JPII).

f has simple poles (linear zeros).

$$\boxed{r = \lim_{z\to z_0} (z-z_0) f(z)}$$ residue.

$$\boxed{\oint_C f(z)\,dz = 2\pi i \cdot \sum r_i}$$ z_i inside C.

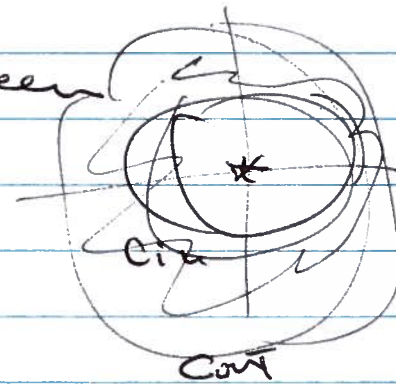
(2)

more general — any negative power.

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n}$$ Laurent series (24.55)

- negative part ("principal part") may be finite.

- convergence may be between two (circles).



(negative $\rightarrow |z-z_0| > R_{in}$
positive $\rightarrow |z-z_0| < R_{out}$

"It may be shown":

analytic between two circles

$\Longleftrightarrow$ (24.55)

$z = z_0 + re^{i\theta}$

$$\oint dz\, f(z) = \int_0^{2\pi} i r e^{i\theta} d\theta \cdot \sum_{(n=-\infty)}^{\infty} a_n (z_0 + re^{i\theta} - z_0)^n$$

$$= \sum_{(n=-\infty)}^{\infty} i a_n r^{n+1} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n+1)\theta}}_{\substack{2\pi \quad n=-1 \\ 0 \quad \text{otherwise}}} = 2\pi i\, a_{-1}$$

(3).

Induction: $a_n = \frac{f^{(n)}(z_0)}{n!}$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

f analytic inside and on C (radius R).

$|f(z)| \le M$ on C.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}.$$

$$\Rightarrow \left|f^{(n)}(z_0)\right| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right| \le \frac{Mn!}{R^n}$$

$$= \frac{n!}{2\pi\cdot R^{n+1}}\left|\oint f(z)\,dz\right| \le \frac{n!}{2\pi R^n}\oint |f(z)|\,dz$$

f analytic and bounded for all z

$n=1$ $R\to\infty$ $|f'(z)| < \frac{M}{R} \to 0.$ $f' = 0$

$\Rightarrow$ $\boxed{f = \text{constant}}$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

evaluate as contour integral.

$\sin \pi z$ has zeroes at $z=n$, but $\sin'$ oscillates in sign

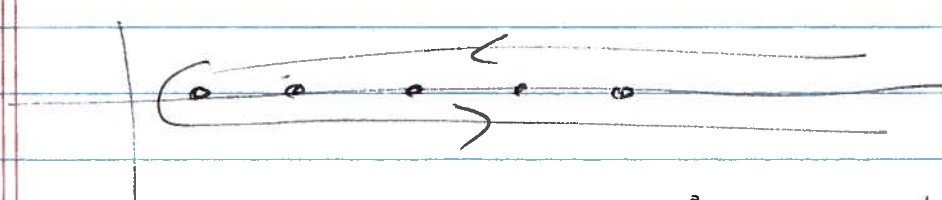
$\tan \pi z$ has zeros at $z=n$

residue $\frac{1}{n^2}$ → look at $\frac{1}{z^2}\frac{1}{\tan \pi z}$

$$r_n = \lim_{z\to n} \frac{(z-n)}{z^2}\frac{1}{\tan \pi z} = \lim_{z'\to 0} \frac{z'}{(n+z')^2}\frac{\cos(n\pi+z'\pi)}{\sin(n\pi+z'\pi)}$$

$$= \lim_{z'\to 0} \frac{z'}{(n^2)}\left(\frac{\cos n\pi \cos z'\pi - \cancel{\sin n\pi}\sin z'\pi}{\cancel{\sin n\pi}\cos z'\pi + \cos n\pi \sin z'\pi}\right)$$

$$= \lim_{z'\to 0}\frac{z'}{n^2}\cdot\frac{\cancel{\cos n\pi}\cdot\cos\pi z'}{\cancel{\cos n\pi}\sin\pi z'} \to \frac{\cancel{z'}\cdot 1}{n^2\cdot\pi\cancel{z'}} = \frac{1}{\pi n^2}$$



$$\oint_C dz \cdot \frac{1}{z^2}\frac{1}{\tan\pi z} = \sum_{n=1}^{\infty}\frac{1}{\cancel{\pi} n^2}\cdot 2\cancel{\pi} i = 2i\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

$2\pi i \cdot \frac{\pi^3}{45} = 4i\,\zeta(4)$

$\zeta(4) = \frac{\pi^4}{90} = 1.08232$

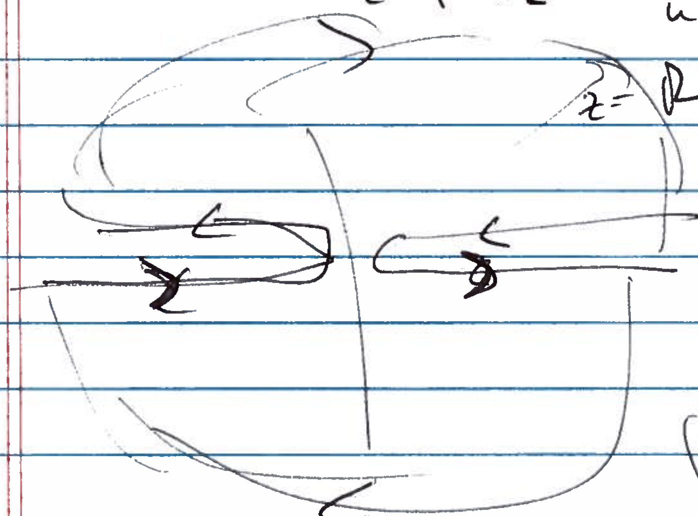
$\frac{1}{\tan \pi z} = \frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \frac{2\pi^5 z^5}{945} + \dots$

not the answer yet.

Negative as 3

C'

$$\oint_{C'} dz \, \frac{1}{z^2} \frac{1}{\tan \pi z} = \sum_{n=-1}^{-\infty} 2\pi i \, \frac{1}{(-n)^2} \frac{1}{\pi} = \text{Same.}$$



$z = Re^{i\theta}$

sin, cos → $e^{iz} \pm e^{-iz}$

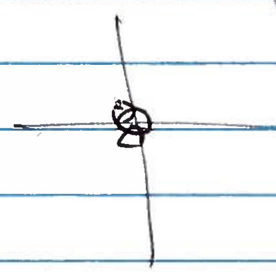
always part is exponentially large → 1

$\int \frac{dz}{|R^2|} \to \frac{1}{R} \to 0.$

Add, complete at ∞.

$$\oint = 4i \sum_n \frac{1}{n^2}$$

deform to origin



$(r \to 0)$ $\oint dz \, \frac{1}{z^2} \frac{1}{\tan \pi z} = \oint dz \left(\frac{1}{\pi z^3} - \frac{\pi}{3z} - \frac{\pi^3 z}{45} + \dots \right)$

$= 2\pi i \left(-\frac{\pi}{3} \right)$

$4i\,\zeta(2) = +\,2\pi i \, \frac{\pi}{3}$

$$\boxed{\zeta(2) = \frac{1}{4} \cdot \frac{2\pi}{3} = \frac{\pi^2}{6}} = 1.64493$$