10|16|2017 "Analytic function" : (all derivatives)

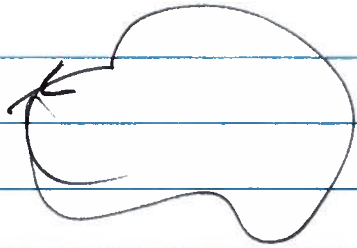
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

$z = x + iy$. $\quad f = u + iv = u(x,y) + i\,v(x,y)$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

linked

$\nabla^2 u = 0 \quad \nabla^2 v = 0 \quad \vec{\nabla} u \cdot \vec{\nabla} v = 0$.



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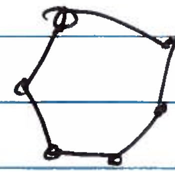


(finite number of corners).

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy)\right]$$

Green's Theorem $\quad \oint_C (P\,dx + Q\,dy) = \int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$

Real part $\quad \left.\begin{array}{l} P = u \\ Q = -v \end{array}\right\} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$

Imaginary part $\quad \left.\begin{array}{l} P = v \\ Q = u \end{array}\right\} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$

$$\boxed{\oint_C f(z)\,dz = 0}$$ for analytic $f(z)$

Cauchy's Theorem

§24.9 (24.40)

$$\boxed{\frac{1}{2\pi i}\oint \frac{f(z')\,dz'}{z'-z_0} = f(z_0)} \qquad (24.46).$$

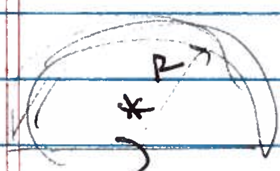
Cauchy's integral formula

§ 24.10

One of the most useful formulas.

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi$$

$x=\tan\theta \qquad dx = \sec^2\theta\, d\theta$



$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

$$\oint \frac{dz}{(z+i)(z-i)} = 2\pi i \cdot f(z)\Big|_{z=i} = 2\pi i \cdot \frac{1}{(i+i)} = \pi$$

$\uparrow f(z) \quad \uparrow z-z_0$

$$R\to\infty \quad \int \frac{iRe^{i\theta}\,d\theta}{(R^2e^{2i\theta}+1)} \longrightarrow O\left(\frac{1}{R}\right) \to 0.$$

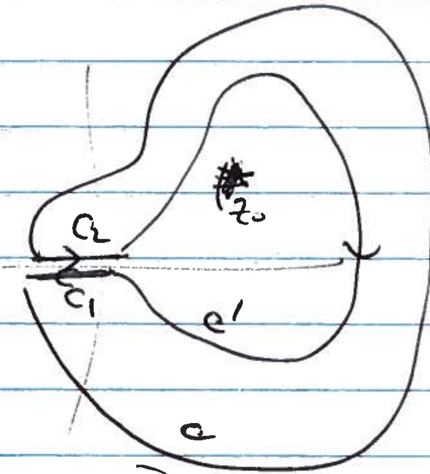
③

Not analytic . $\frac{f(z)}{z-z_0}$

(f(z) analytic)

$$\oint_{c_1+C+c_2+C'} \frac{f(z)}{z-z_0}\,dz = 0.$$

(also w/o pole)

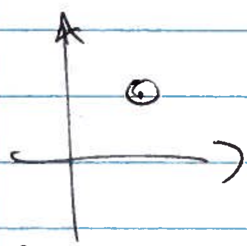


c_1 cancels c_2 c, c' opposite directions

ccw. $\boxed{\oint_c \frac{f(z)}{z-z_0}\,dz = \oint_{c'} \frac{f(z)}{z-z_0}\,dz.}$

As long as c' wraps z_0, deformation doesn't cross z_0.

Deform to small circle of radius r around $z=z_0$. $z = z_0 + re^{i\theta}$



$$\oint dz = \int_0^{2\pi} d(re^{i\theta}) = ire^{i\theta}\,d\theta$$

$$\oint \frac{f(z)}{z-z_0}\,dz = \oint \frac{f(z)\, ire^{i\theta}\,d\theta}{(\cancel{z_0}+re^{i\theta})-\cancel{z_0}} = \oint f(z)\, i\,d\theta$$

$$= 2\pi i \cdot f(z_0)$$

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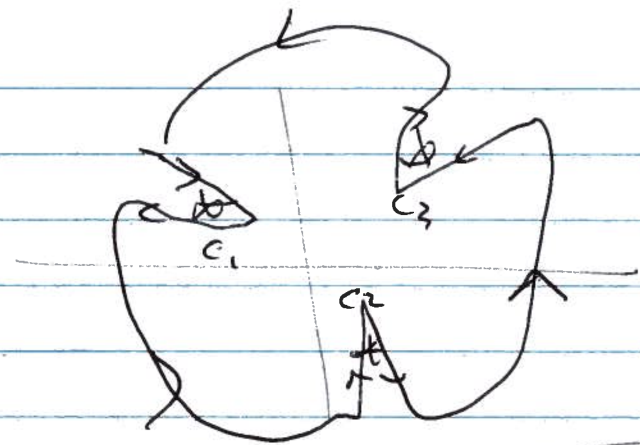
multiple Poles.



$$\oint_C \longrightarrow \oint_{c_1} + \oint_{c_2} + \oint_{c_3} \rightarrow \boxed{2\pi i \cdot \sum r_i}$$

$\frac{1}{\sin x}$ doesn't factor, exactly.

$$\frac{1}{\sin x} = \frac{1}{x}\left(\frac{x}{\sin x}\right)$$

↑ pole ↖ analytic @ $x=0$.

$$\frac{x}{\sin x} = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \frac{127}{604800}x^8$$

$$+ \frac{73}{3421440}x^{10} + \frac{1414477}{653837184000}x^{12} + \cdots$$

$1414477 = 23 \cdot 89 \cdot 691$