10/13/17 Complex series (arithmetic)

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin n\phi = \text{Im} \sum_n \frac{1}{n} e^{in\phi} = \text{Im} \sum \frac{1}{n} \left(e^{i\phi}\right)^n \qquad z = e^{i\phi}$$

$$= \text{Im} \sum_n \frac{1}{n} z^n = \text{Im}\left(-\ln(1-z)\right)$$

$$1-e^{i\phi} = (1-\cos\phi) - i\sin\phi = r e^{i\theta}$$

$$r^2 = (1-\cos\phi)^2 + (\sin\phi)^2 = 1 - 2\cos\phi + \cos^2\phi + \sin^2\phi = 2 - 2\cos\phi$$

$$= 4\left(\tfrac{1}{2}(1-\cos\phi)\right) = 4\sin^2\frac{\phi}{2} \qquad \boxed{r = 2\sin\frac{\phi}{2}}$$

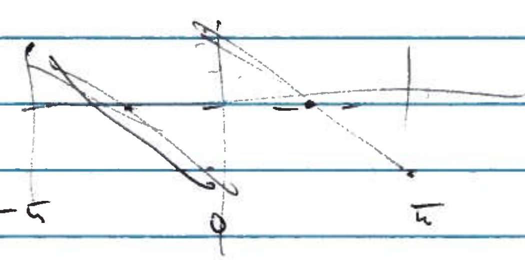
$$\tan\theta = -\frac{\sin\phi}{1-\cos\phi} = -\frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}} = -\frac{\cos\phi/2}{\sin\phi/2}$$

$$\tan\left(x-\frac{\pi}{2}\right) = \frac{\sin\left(x-\frac{\pi}{2}\right)}{\cos\left(x-\frac{\pi}{2}\right)} = \frac{\sin x \cancel{\cos\frac{\pi}{2}} - \cos x \sin\frac{\pi}{2}}{\cos x \cancel{\cos\frac{\pi}{2}} + \sin x \sin\frac{\pi}{2}} = -\frac{\cos x}{\sin x}$$

$$\tan\theta = -\frac{\cos\phi/2}{\sin\phi/2} \qquad \theta = \frac{\phi}{2} - \frac{\pi}{2}$$

$$\text{Im}\left(-\ln\left(r e^{i\theta}\right)\right) = \text{Im}\left(-\ln r - i\theta\right) = -\theta = \frac{\phi}{2} - \frac{\pi}{2}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \sin n\phi = \frac{\pi}{2} - \frac{\phi}{2}}$$



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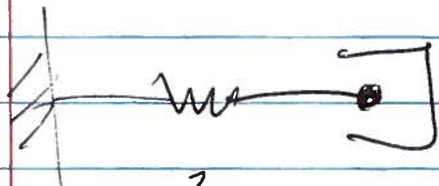


$F = -kx - bv + F_{ext}$

$$m\frac{d^2x}{dt^2} + \gamma m \frac{dx}{dt} + m\omega_0^2 x = F_{ext} = F_0 \cos\omega t$$

$$= \mathrm{Re}\{F_0 e^{-i\omega t}\}$$

<u>Assume:</u> $x = x_0 e^{-i\omega t}$ (Re).

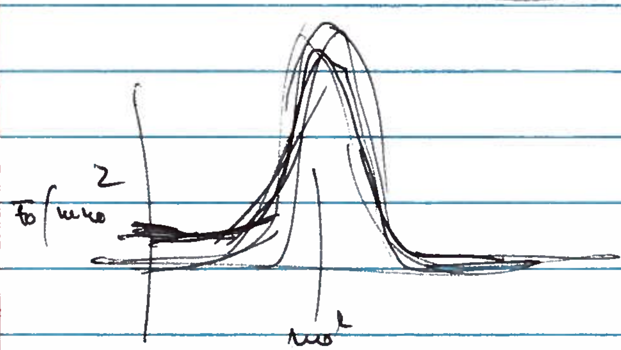
$x' = -\omega x \qquad x'' = -\omega^2 x$

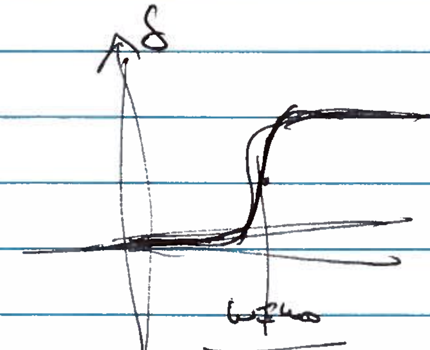
$$-m\omega^2 x_0 + \gamma m(-i\omega x_0) + m\omega_0^2 x_0 = F_0.$$

$$x_0\left(\omega_0^2 - \omega^2 - i\gamma\omega\right) = F_0/m.$$

$x_0 = |x_0| e^{i\delta}$ $\qquad x_0 = \dfrac{F_0/m}{(\omega_0^2-\omega^2) - i\gamma\omega}.$

$$|x_0| = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2\right]^{1/2}} \qquad \tan\delta = \frac{\gamma\omega}{\omega_0^2-\omega^2}$$





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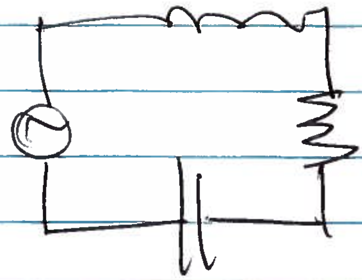

$V = V_0 \cdot \cos \omega t$

$= L\frac{dI}{dt} + IR + \frac{Q}{C}$

$L\frac{d^2I}{dt^2} + R\cdot\frac{dI}{dt} + \frac{1}{C}\cdot I = \frac{dV}{dt}$

$V = V_0 e^{-i\omega t}$

$e^{j\omega t}$

$I = i$

$I = I_0 e^{-i\omega t}$

$-I_0 L \omega^2 - i\omega R I_0 + \frac{I_0}{C} = -i\omega V_0$

$I_0\left(-i\omega I_0 L + R + \frac{1}{-i\omega C}\right) = V_0$

$$I_0 = \frac{V_0}{R - i\left(\omega L - \frac{1}{\omega C}\right)} = \frac{V_0}{Z}.$$

$$|I_0| = \frac{|V_0|}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}}$$

Functions   Chapter 24

$z = x + iy$

$$f(z) = u + iz = u(x,y) + i\,v(x,y)$$

$$z^2 = (x+iy)^2 = (x^2 - y^2) + 2ixy$$

$$e^{iz} = e^{i(x+iy)} = e^{ix} e^{-y} = (\cos x + i \sin x)\, e^{-y}$$

$$= \cos x\, e^{-y} + i \cdot \sin x\, e^{-y},$$

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

$$= \frac{\cancel{\Delta u + i\Delta v}}{\Delta x + i\Delta y}$$

"Analytic", "regular", "holomorphic"

("meromorphic") isolated poles

("monogenic")

"analytic" at P $\Leftrightarrow$ analytic w/in $E \ni P$

(1) approach along $\Delta y = 0$

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \left(\frac{\Delta u + i \Delta v}{\Delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

(2) approach along $\Delta x = 0$

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \left(\frac{\Delta u + i \Delta v}{i \Delta y} \right) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Well defined ⇒ these are the same

⇒ real, imag. parts the same.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \qquad (24.5)$$

Cauchy-Riemann relations

$z^* = x - iy$ $\quad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = -1$ ✗.

"regular point" → neighborhood.

not regular → "singularity"

isolated, or worse.

(6)

Satisfied $\Longleftrightarrow$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \to 0$$

Any analytic $f(z)$ is solution (z) to Laplace's equation.

e^{iz} $\quad u = \cos x \, e^{-y} \qquad (-\cos x)e^{-y} + \cos x(+e^{-y}) = 0.$

$$\vec{\nabla} u \cdot \vec{\nabla} v = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial x}\right)\left(-\frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right) = 0$$

potential flow. $\vec{v} = -\vec{\nabla} u$

follows: $V = \text{constant}$

$f = z = x + iy \qquad u = x \qquad E = \hat{x}$

$v = y = \text{constant}$

$f = \ln z = \rho + i\phi \qquad E = \hat{\rho} \qquad \phi = \text{constant}$

$$z = (\rho e^{i\phi}) = r e^{i\theta} \qquad f(z)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} = \frac{df}{dz} e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial \theta} = \frac{df}{dz} \, i r e^{i\theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$\Rightarrow e^{i\theta}\frac{df}{dz} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \qquad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

f "<u>analytic</u>" $\Rightarrow$ <u>All</u> derivatives well-defined, unique.

$\hookrightarrow$ Taylor series.

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{1}{2} f''(z_0)(z-z_0)^2 + \cdots$$

Radius of convergence = distance to nearest singular point