10 | 11 | 17.

Chapter 3

integers $\mathbb{Z}$ - cardinality of sets

$\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$

addition ⟺ unity subtraction (inverse)

multiplication, primes "ring" $j-2=0$

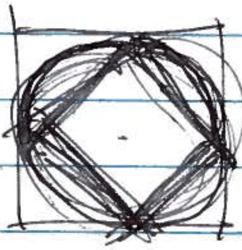
rational $\mathbb{Q} = \{(a,b)\}$ "field"

$(a,b) + (c,d) = (ad+bc, bd)$.

$(a,b) * (c,d) = (ac, bd)$ $3q+2=0$

real $\mathbb{R}$. cauchy sequences. $\sqrt{2}$, π

$x^2-2=0$



(r=1)

$2\sqrt{2} = 2.828$

$4\sqrt{2} < 2\pi < 8$ circumference

$(2) < \pi < (4)$ area

Complex $\mathbb{C}$ $z = x+iy$.

$(a+ib)(c+id) = (ac-bd) + i(bc+ad)$

$i^2 = -1$ $z^2+2=0$

$0.99999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} = \sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{.9}{1-0.1} = 1$

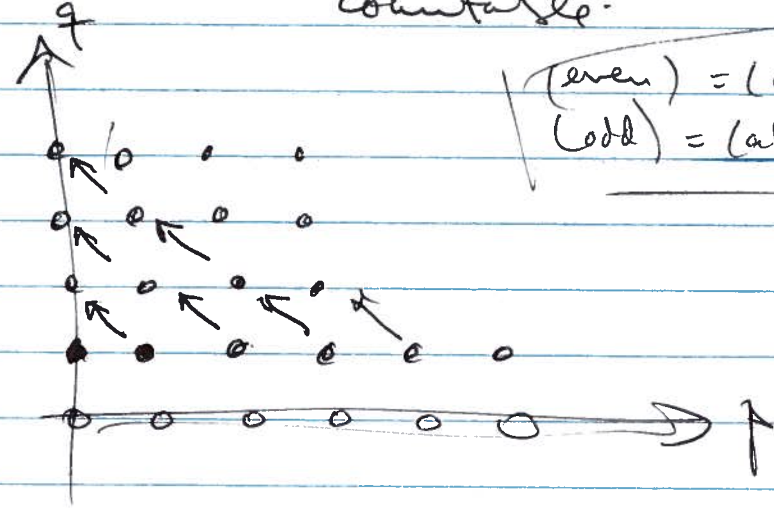
$|a_n - 1| = 10^{-n} \to 0$.

$\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$ infinite (always another)

"countable"

(even) = (all)
(odd) = (all)

$Q = (p, q)$



$(0,1) = 0$
$(1,1) = 1$
~~$(0,2) = 0$~~

$(2,1) = 2$
$(1,2) = \frac{1}{2}$
~~$(0,3) = 0$~~

$(3,1) = 3$
~~$(2,2) = 1$~~
$(1,3) = \frac{1}{3}$
~~$(0,4) = 0$~~

rationals = # integers "countable" $\aleph_0$

reals $0 < x < 1$ $x = \{0.d_1 d_2 d_3 d_4 \ldots\}$

suppose: all x's ordered. $\{x_1, x_2 \ldots\}$

construct another x: $d_1 \neq d_{1,1}$
$d_2 \neq d_{2,2}$
$d_3 \neq d_{3,3}$

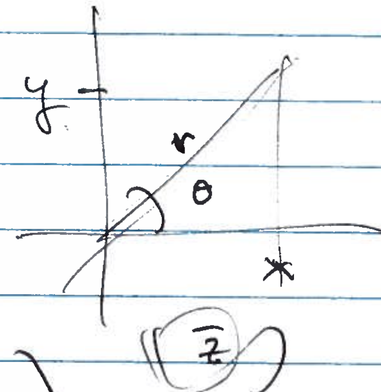
"$\aleph_1 = 2^{\aleph_0}$"

$x \neq x_1, x \neq x_2, x \neq x_3 \ldots$ contradiction

Complex: not countable, not orderable

2

$z = x + iy \rightarrow$ point in plane



$\underline{z = r e^{i\theta}}$

$r = |z| = \sqrt{x^2+y^2} = \sqrt{z z^*}$

$\tan\theta = \frac{y}{x}$ (4 quadrants)

$\mathrm{Re}\{z\} = x = \mathrm{Re}\, z^*$

$\mathrm{Im}\{z\} = y. = -\mathrm{Im}\, z^*$

$\boxed{\text{Im } z \text{ is real}}$

$(1+i)^2 = 1 + 2i + i^2 = \underline{\underline{2i}}$

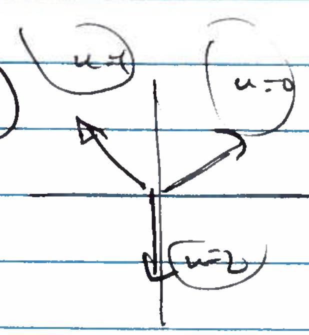
$\sqrt{i} = \pm \frac{(1+i)}{\sqrt{2}}$

$\frac{1}{1+i} = \frac{1}{1+i}\frac{(1-i)}{(1-i)} = \frac{1-i}{1+\cancel{i}-\cancel{i}-i^2} = \frac{1-i}{2}$

$i^{1/3} = ??$ $\quad i = 0 + i\cdot 1 = (0,1)$ $\quad r = 1 \quad \theta = \frac{\pi}{2}$

$i = e^{i\pi/2} \qquad i = e^{i(\frac{\pi}{2} + 2n\pi)}$

$i^{1/3} = \left[e^{i(\frac{\pi}{2}+2n\pi)}\right]^{1/3} = e^{i(\frac{\pi}{6} + \frac{2\pi}{3}n)}$



$n=0 \quad e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$

$n=1 \quad e^{i\frac{5\pi}{6}} = \cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$

$n=2 \quad e^{i\frac{3\pi}{2}} = \cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}) = 0 + i(-1) = -i$

$(\frac{\sqrt{3}}{2} + i\frac{1}{2})^3 = (\frac{\sqrt{3}}{2})^3 + 3(\frac{\sqrt{3}}{2})^2(\frac{i}{2}) + 3(\frac{\sqrt{3}}{2})(\frac{i}{2})^2 + (\frac{i}{2})^3$

Series (1.) 3.

$z = \frac{1+i}{2}$ $\qquad \sum_{n=0}^{\infty} z^n = 1 + \left(\frac{1+i}{2}\right) + \left(\frac{1+i}{2}\right)^2 + \cdots$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{1+i}{2}\right)^{n+1}}{\left(\frac{1+i}{2}\right)^{n}}\right| = \left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1.$$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} = \frac{1}{1-\left(\frac{1+i}{2}\right)} = \frac{2}{2-(1+i)} = \frac{2}{1-i}$$

$$= \frac{2\,(1+i)}{(1-i)(1+i)} = \frac{2+2i}{1-i+i-i^2} = \underline{1+i}$$

$$1 + \left(\frac{1+i}{2}\right) + \left(\frac{\cancel{1}+2i+\cancel{i^2}}{4}\right) + \left(\frac{1+3i+3i^2+i^3}{8}\right) + \left(\frac{1+4i+6i^2+4i^3+i^4}{16}\right)$$

$$= 1 + \left(\frac12 + \frac{i}{2}\right) + \left(\frac{i}{2}\right) + \left(-\frac14 + \frac{i}{4}\right) + \cancel{\left(-\frac18 - \frac{i}{8}\right)}$$

$$+ \left(-\frac14\right) + \left(-\frac18 - \frac{i}{8}\right) + \left(\frac{-i}{8}\right) + \cdots$$

$$= \left(1 + \cancel{\frac12} - \cancel{\frac14} - \cancel{\frac14}\right) + i\left(\frac12 + \frac12 + \frac14 - \frac18\right)$$

$-\frac18$ $\qquad + \cdots$

$\sum 1/\sqrt{n}$ doesn't converge. But

$$\sum i^{n-1}/\sqrt{n} = 1 + \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{i}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \frac{i}{\sqrt{6}} - \frac{1}{\sqrt{7}} - \frac{i}{\sqrt{8}}$$

$$= \left(1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \dots\right) + i\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \dots\right)$$

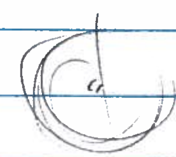
$$= \mathrm{Polylog}\left(\tfrac{1}{2}, i\right) = -0.427728 + 0.667691\,i$$

$$* \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\zeta\left(\tfrac{1}{2}\right)$$

Ratio test "Absolute convergence"

$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z| < 1.$$

Always a circle



$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2} z^2 + \frac{1}{6} z^3 + \dots = e^z$$

$$\left|\frac{a_{n+1}}{a_n}\right| |z| = \frac{|z|}{n+1} \quad (n \to \infty) \quad < 1 \text{ always.}$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta)^n = \sum_{n \text{ even}} \frac{1}{n!} \underbrace{i^n}_{\pm 1} \theta^n + \sum_{n \text{ odd}} \frac{1}{n!} \underbrace{i^n}_{\pm i} \theta^n$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} (-1)^k \theta^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} i(-1)^k \theta^{2k+1}$$

$$= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots\right) + i\left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^3 + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$ Euler's formula.

$$\boxed{1 + e^{i\pi} = 0}$$

"If not immediately obvious, not a first class mathematician"

"How many mathematicians does it take to change a light bulb" $\boxed{-e^{i\pi}}$