10/9/2017

$$\boxed{\int dx\, f(x)\, \delta(x) = f(0).} \quad (13.12)$$

$$\boxed{\delta(ax) = \frac{1}{|a|}\delta(x)} \quad (13.16)$$

$$\cancel{\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)}$$

$$\delta(\vec{r}-\vec{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0) \quad (13.21)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\sqrt{2\pi}\,\varepsilon} \exp\left(-\frac{(x-x_0)^2}{2\varepsilon^2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y-y_0)^2}{2\varepsilon^2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(z-z_0)^2}{2\varepsilon^2}\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^3 \exp\left(-\frac{1}{2\varepsilon^2}\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)\right)$$

$$\Delta x^2 + \Delta y^2 + \Delta z^2$$

Lean

$$\Delta r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

$$\rightarrow (r-r_0)^2 + r^2(\theta-\theta_0)^2 + r^2\sin^2\theta\,(\phi-\phi_0)^2$$

$$\Delta r^2 + r^2\Delta\theta^2 + r^2\sin^2\theta\,\Delta\phi^2$$

$$\delta(\vec{r}-\vec{r}_0) = \left(\frac{1}{\sqrt{2\pi}}\frac{1}{\varepsilon}\right)^3 \exp\left(-\frac{(r-r_0)^2}{2\varepsilon^2}\right) \exp\left(-\frac{r^2(\theta-\theta_0)^2}{2\varepsilon^2}\right) \exp\left(-\frac{(\phi-\phi_0)^2 r^2\sin^2\theta}{2\varepsilon^2}\right)$$

$$= \delta(r-r_0)\;\delta(r(\theta-\theta_0))\;\delta(r\sin\theta(\phi-\phi_0))$$

$$= \frac{1}{r^2\sin\theta}\,\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)$$

Jacobian.

$$\int d^3V \cdot f(r)\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^3}\right) \to \int_{(r<a)} d^3r\, f\, \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

(any volume containing $r=0$)

$$= \int_{r<a} d^3r \left[\vec{\nabla}\cdot\left(\frac{f\hat{r}}{r^2}\right) - \frac{\hat{r}}{r^2}\cdot\vec{\nabla} f \right)$$

$$= \oint_{r=a} d^2S\, \hat{r}\cdot\left[\frac{f\hat{r}}{r^2}\right) - \int_{r<a} d^3r\, \frac{\hat{r}}{r^2}\cdot\hat{r}\frac{\partial f}{\partial r}$$

$$= \oint_{r=a} \overbrace{d^2S}^{4\pi r^2}\, \frac{f(r)}{r^2} - \int 4\pi \cancel{r^2} dr\, \frac{1}{\cancel{r^2}}\frac{\partial f}{\partial r}$$

$$= \underline{4\pi \cancel{a^2}\cdot\frac{f(a)}{\cancel{a^2}}} - 4\pi \underbrace{\int_0^a \frac{\partial f}{\partial r}\, dr}_{f(a)-f(0)}$$

$$= 4\pi f(a) - \left(4\pi f(a) - 4\pi f(0)\right)$$

$$= 4\pi\, f(0)$$

any sphere containing ($r=0$)

Hint

$$\textcircled{2}$$

$$\left\{ \begin{array}{l} \vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \cancel{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right)} = 0. \quad (r \neq 0) \\ \int d^2a\,\hat{n}\cdot\vec{E} = Q/\varepsilon_0 \end{array} \right.$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,\hat{r}}{r^2} \to \frac{1}{4\pi\varepsilon_0}\frac{Q\,\vec{r}}{(r^2+\varepsilon^2)^{3/2}}$$

$$= -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{r^2+\varepsilon^2}}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\varepsilon_0}\frac{r^3}{(r^2+\varepsilon^2)^{3/2}}\right)$$

$$= \frac{1}{r^2}\left(\frac{3r^2}{(r^2+\varepsilon^2)^{3/2}}\frac{(r^2+\varepsilon^2)}{(r^2+\varepsilon^2)} - \frac{3r^3\cdot r}{(r^2+\varepsilon^2)^{5/2}}\right)$$

$$= \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}\cdot\frac{Q}{4\pi\varepsilon_0} = \rho/\varepsilon_0$$

$$\boxed{\rho = Q\cdot\frac{3\varepsilon^2}{4\pi(r^2+\varepsilon^2)^{5/2}}}$$

③

$$\frac{3}{4\pi}\frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}$$

$\varepsilon \to 0$ $\quad$ $r \neq 0$ $\quad \frac{3}{4\pi}\frac{\varepsilon^2}{r^5} \to 0$

$r = 0$, $\quad \frac{3}{4\pi}\frac{\varepsilon^2}{\varepsilon^5} \to \infty$

$$\int d^3r \cdot \frac{3}{4\pi}\frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = \int \cancel{4\pi}\, r^2\,dr \cdot \frac{3}{\cancel{4\pi}}\frac{\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}$$

$r = \varepsilon \cdot \tan\theta \qquad dr = \varepsilon \cdot \sec^2\theta\, d\theta$

$$= 3\varepsilon^2 \int_0^{\pi/2} \frac{(\varepsilon \tan\theta)^2(\varepsilon \sec^2\theta\, d\theta)}{(\varepsilon^2+\varepsilon^2\tan^2\theta)^{5/2}}$$

$$= 3\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta\, d\theta}{\sec^5\theta}$$

$$= 3\int_0^{\pi/2} \frac{\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos^2\theta}}{\frac{1}{\cos^5\theta}}\, d\theta$$

$\cos\theta \sin^2\theta\, d\theta = \frac{d}{d\theta}\left(\frac{\sin^3\theta}{3}\right)$

$$= 3\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2} = 1 \checkmark$$

$$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^{(3)}(\vec{r})}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} = \frac{-\hat{r}}{r^2} \qquad \boxed{\nabla^2\frac{1}{r} = -4\pi\,\delta^{(3)}(\vec{r})}$$

$$\oint_S \vec{E}\cdot\hat{n}\,d^2a = \frac{Q}{\varepsilon_0} = \int_V d^3r\,\frac{\rho(r)}{\varepsilon_0} = \int d^3r\,(\vec{\nabla}\cdot\vec{E})$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \qquad \rho_Q = Q\,\delta^{(3)}(\vec{r})$$

$$\oint_S \vec{B}\cdot d\vec{r} = \mu_0 I_\perp = \mu_0\int_S \vec{J}\cdot\hat{n}\,d^2a = \int_S (\vec{\nabla}\times\vec{B})\cdot\hat{n}\,d^2a.$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{j}}$$

$$\mathcal{E} = \oint_C \vec{E}\cdot d\vec{r} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\,d^2a$$

$$= -\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,d^2a = \int_C (\vec{\nabla}\times\vec{E})\cdot\hat{n}\,d^2a$$

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Displacement Current: $I_d = \frac{\partial}{\partial t}\int d^2a\,\hat{n}\cdot(\varepsilon_0 \vec{E})$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 (I + I_d)$$

$$= \mu_0 \int_S d^2a\,\hat{n}\cdot\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}$$ Maxwell.

Implications

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}$$

$$\frac{\partial \rho}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E}) = \varepsilon_0 \vec{\nabla}\cdot\left(\frac{\partial \vec{E}}{\partial t}\right)$$

$$= \varepsilon_0 \vec{\nabla}\cdot\left(\frac{1}{\mu_0\varepsilon_0}\left((\vec{\nabla}\times\vec{B}) - \mu_0\vec{j}\right)\right) = -\vec{\nabla}\cdot\vec{j}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$$ Maxwell $\Rightarrow$ charge conservation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\rho$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 \vec{E}$$

$$= -\vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$= -\frac{\partial}{\partial t}\left(\cancel{\mu_0 \vec{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0}$$

$$c^2 = \frac{1}{\mu_0 \epsilon_0} = \frac{4\pi}{\mu_0} \cdot \frac{1}{4\pi\epsilon_0} = (10^7)\,(9 \cdot 10^9) = (3 \cdot 10^8)^2 \ m^2/s^2$$

$$9 = 8.98755178 = (2.99792458)^2$$

$\vec{\nabla}\cdot\vec{B} = 0$ $\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ vector potential

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right) = 0$

$\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\Phi$ $\boxed{\vec{E} = -\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}}$

Given. $\vec{\nabla}\times\vec{A} = \vec{B}$ $\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$ $\vec{\nabla}\times\vec{A}' = \vec{B}$ also.

$\Phi' = \Phi - \frac{\partial\Lambda}{\partial t}$

$\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial\vec{A}'}{\partial t} = -\vec{\nabla}\left(\Phi - \frac{\partial\Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A}+\vec{\nabla}\Lambda\right)$

$= \vec{E} + \vec{\nabla}\frac{\partial\Lambda}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Lambda = \vec{E}$

gauge invariance, gauge transformation

QM : $\vec{p} \to \underbrace{-i\hbar\vec{\nabla}} \to (\vec{p} - e\vec{A})$

(free particle)

$$\underline{\psi' = e^{ie\Lambda/\hbar}}$$

$$(\vec{p} - e\vec{A}')\psi' = (-i\hbar\vec{\nabla} - e\vec{A} - e\vec{\nabla}\Lambda)\left(e^{ie\Lambda/\hbar}\psi\right)$$

$$= \Big(-i\hbar\vec{\nabla}\psi \cdot e^{\frac{ie\Lambda}{\hbar}} + \cancel{\psi(-i\hbar\vec{\nabla}\left(\tfrac{ie\Lambda}{\hbar}\right)e^{\frac{ie\Lambda}{\hbar}})}$$

$$- e\vec{A}\, e^{ie\Lambda/\hbar}\psi - \cancel{e\vec{\nabla}\Lambda\, e^{\frac{ie\Lambda}{\hbar}}\psi}\Big)$$

$$= \underline{e^{\frac{ie\Lambda}{\hbar}}(\vec{p} - e\vec{A})\psi}.$$

$$\boxed{|(\vec{p} - e\vec{A})\psi|^2 \quad \underline{\text{invariant}}}$$