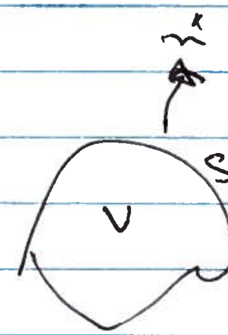
10/4/17 Important Theorems

(11.18) $$\boxed{\int_V \vec{\nabla}\cdot\vec{a}\, d^3V = \oint_S \vec{a}\cdot d\vec{S}}$$

$$\boxed{\int_V \vec{\nabla}\cdot\vec{A}\, d^3V = \oint_S \vec{A}\cdot\hat{n}\, d^2S}$$



outward.

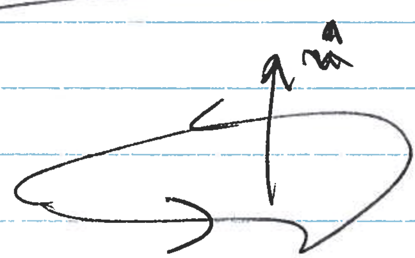
closed.

divergence Theorem

(11.23) $$\boxed{\int_S (\vec{\nabla}\times\vec{a})\cdot d\vec{S} = \oint_C \vec{a}\cdot d\vec{r}}$$

$$\boxed{\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, d^2S = \oint_C \vec{A}\cdot d\vec{r}}$$

closed.



RHR.

②

$\vec{E} = -\vec{\nabla} V$ $\qquad$ $\vec{\nabla} \times \vec{E} = 0$

no curl, only div → "twistles"

$\vec{B} = \vec{\nabla} \times \vec{A}$ $\qquad$ $\vec{\nabla} \cdot \vec{B} = 0$

no div only curl → wraps

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla}) V = -\nabla^2 V = \rho/\varepsilon_0$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{A}}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Stokes: $\vec{a} = \phi \vec{c}$ $\qquad$ $\vec{c}$ = constant

$$\int_S \vec{\nabla} \times (\phi \vec{c}) \cdot \hat{n}\, d^2S = \int_C (\phi \vec{c}) \cdot d\vec{r} =$$

$$\int_S (\vec{\nabla}\phi) \times \vec{c} \cdot \hat{n} = \int_S \vec{c} \cdot (\hat{n} \times \vec{\nabla}\phi)\, d^2S$$

$$\varepsilon_{ijk} \nabla_i c_j \hat{n}_k = \varepsilon_{ijk} c_i \hat{n}_j \nabla_k$$

$$\vec{c} \cdot \int_S (\hat{n} \times \vec{\nabla}\phi)\, d^2S = \vec{c} \cdot \oint_C \phi\, d\vec{r}$$

$$\boxed{\int_S (\hat{n} \times \vec{\nabla}\phi)\, d^2S = \oint_C \phi\, d\vec{r}} \quad (11.24)$$

Issue : point Q.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q\hat{r}}{r^2}$$

Table 10.3

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\cancel{\left(\sin\theta E_\theta\right)} + \frac{1}{r\sin\theta}\cancel{\frac{\partial E_\phi}{\partial\phi}}.$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_0}Q\right) = 0$$

But.
$$\oint_{\text{sphere}(r)} d^2S\,\hat{n}\cdot\vec{E} = \oint d^2S.\ \hat{r}\cdot\vec{E} = 4\pi r^2\cdot\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)$$

$$= Q/\epsilon_0$$

$$\int_V d^3V\,\vec{\nabla}\cdot\vec{E} = \oint_S \vec{E}\cdot\hat{n}\,d^2S \quad ??$$

$\vec{\nabla}\cdot\vec{E} = 0$ except at $r = 0$

Dirac δ-function. $\boxed{\S 13.1.3}$

idealization. $\delta(x) = \begin{cases} 0 & x \neq 0, \\ \infty & x = 0, \end{cases}$ (13.11)

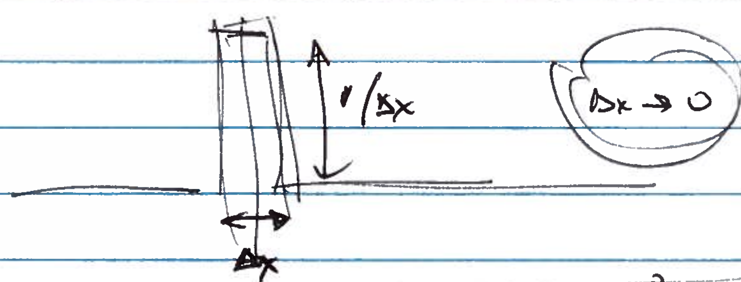
such that $\int_a^b dx\, \delta(x) = 1$ $\begin{cases} a<0 \\ b>0 \end{cases}$ (13.12)

(as small as you like).

or: $\int dx\, f(x)\, \delta(x) = f(0)$.

Can't really do that. $\longrightarrow$

limit of something $\begin{cases} \text{taller \& taller} \\ \text{narrower \& narrower.} \end{cases}$



Nice: $\frac{1}{\sqrt{2\pi}}\frac{1}{(\Delta x)} e^{-\frac{1}{2}\frac{x^2}{(\Delta x)^2}}$ Gaussian. (smooth, integrable)

$\Delta x \to 0$.

Properties

$$[\delta(x)] = [x]^{-1}$$
$$[\delta(t)] = [t]^{-1}.$$

$$\int dx\, f(x)\, \delta(x-a) = \,?!$$

$x' = x - a$
$x = x' + a$
$dx = dx'$

$$= \int dx'\, f(x'+a)\, \delta(x') = f(0+a)$$

$$\boxed{\int dx\, f(x)\, \delta(x-a) = f(a)} \qquad (13.12)$$

(sometimes taken as defining property)

$$\int dx\, f(x)\, \delta(ax) = \int \frac{dx'}{|a|} f\left(\frac{x'}{a}\right) \delta(x') = \frac{1}{|a|} f(0)$$

$x' = ax$
$x = \frac{x'}{a}$

$$\boxed{\delta(ax) = \frac{1}{|a|}\delta(x)} \qquad (13.16)$$

$a = -1$ $\qquad \boxed{\delta(-x) = \delta(x)}$ $\qquad$ **even** function (13.15).

$$\int_a^b dx\, f(x)\, \delta'(x) = \Big[f(x)\delta(x)\Big]_a^b - \int dx\, f'(x)\, \delta(x).$$

$$\boxed{\int dx\, f(x)\, \delta'(x) = -f'(0)}$$

$$\boxed{\delta'(-x) = -\delta'(x)}$$

$$x\,\delta(x) = \;?$$

$$\int dx\; f(x)\, x\,\delta(x) = \left(x f(x)\right)\Big|_{x=0} = 0.$$

$$\boxed{x\,\delta(x) = 0} \qquad (13.17).$$

$$\int dx\; f(x)\, \delta(h(x))$$

let $x' = h(x)$

$dx' = h'(x)\,dx$

$$= \int \frac{dx'}{|h'(x)|}\, f\left(h^{-1}(x')\right)\, \delta(x')$$

$\underline{h(x) = 0}$ at $x = x_0$.

$$\rightarrow \frac{1}{|h'(x_0)|}\, f(x_0)$$

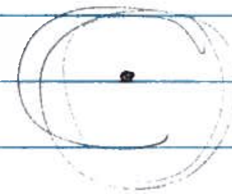
$$\rightarrow \sum_i \frac{1}{|h'(x_i)|}\, f(x_i)$$

$$\boxed{\delta(h(x)) = \sum_i \frac{1}{|h'(x_i)|}\, \delta(x - x_i)}$$

near $x = x_i$ $\quad h = \cancel{h(x_i)} + \underline{h'(x_i)\,(x - x_i)} + \ldots$

$$\delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

Point charge



$$\int \hat{E} \cdot \hat{n}\, d^2a = 4\pi r^2 \cdot E = Q/\epsilon_0$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \cancel{\frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, E_\theta\right)} + \cancel{\frac{1}{r \sin\theta}\frac{\partial E_\phi}{\partial \phi}}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi\epsilon_0}\right) = 0$$

Flux $\neq 0$ divergence $= 0$

$\boxed{\delta\text{-function}}$