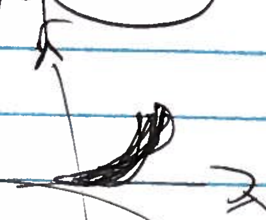
9/20/17 lengths and areas. curves and surfaces

length $ds^2 = dx^2 + dy^2 = dx^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$

$ds = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$, $\S(0,1)$ $(0,1)$



parabola. $y = x^2$ $\quad \frac{dy}{dx} = 2x$

$s(0\to1)$: greater than 1 ~~less~~ more than $\sqrt{2}$

$s(0,1) = \int_0^1 ds = \int_0^1 dx\sqrt{1+2x^2}$

$1+(\text{something})^2$ $\quad \sin^2\theta+\cos^2\theta = 1$

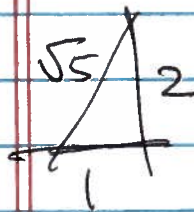
$\tan^2\theta + 1 = \sec^2\theta$

let $4x^2 = \tan^2\theta$ $\quad \tan\theta = 2x$ $\quad \theta_0 = 1$

$x = \frac{1}{2}\tan\theta$ $\quad \tan\theta_1 = 2$

$$S = \int_0^{\theta_1}\left(\tfrac{1}{2}\sec^2\theta\, d\theta\right)\sqrt{1+\tan^2\theta} = \frac{1}{2}\int_0^{\theta_1}\sec^3\theta\, d\theta$$

$$= \frac{1}{4}\Big[\sec\theta\tan\theta + \log(\sec\theta+\tan\theta)\Big]_0^{\theta_1}$$



$\tan\theta_1 = 2$

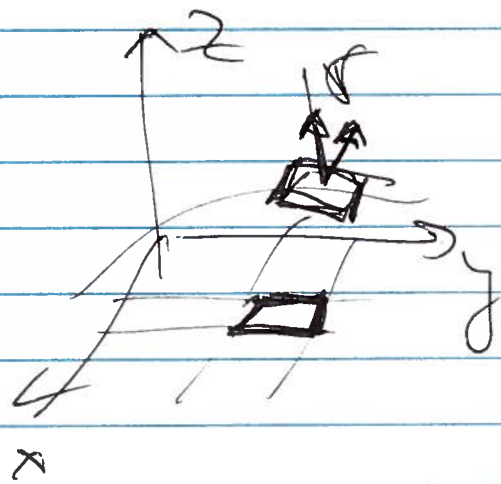
$\cos\theta_1 = 1/\sqrt{5}$ $\qquad \sec\theta_1 = \sqrt{5}$.

$$S = \frac{1}{4}\left(\sqrt{5}\cdot 2 + \underbrace{\log\left(2+\sqrt{5}\right)}_{\sinh^{-1}\sqrt{5}} \right)$$

$$= 1.47894$$

Surface

$z = z(x,y)$



tilted $\quad$ area in plane $= dA_\perp = dx\,dy$

$$dA_\perp = dA \cdot \cos\gamma \qquad dA = \frac{dA_\perp}{\cos\gamma}$$

As constraint: $\phi(x,y,z) = 0$

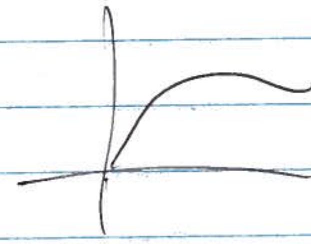
$$d\phi = \left(\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial\phi}{\partial z}\right)dz.$$

$$= (\vec{\nabla}\phi)\cdot(d\vec{x}) = 0.$$

3

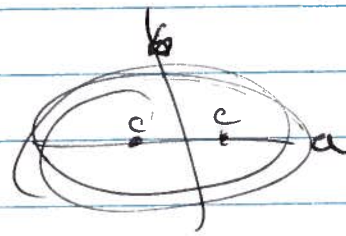
$x = x(\theta)$

$y = y(\theta)$



$$S = \int \sqrt{dx^2 + dy^2} = \int d\theta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

Ellipse:



$x = a \cdot \cos\theta$

$y = b \cdot \sin\theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

$$= a \sqrt{\sin^2\theta + \cos^2\theta - \frac{(a^2 - b^2)}{a^2} \cos^2\theta}$$

$$\boxed{S = a \int_0^{2\pi} d\theta \sqrt{1 - \frac{c^2}{a^2} \cos^2\theta}}$$

$$\boxed{s = 4a\, E\left(\frac{c^2}{a^2}\right)}$$ Elliptic integral

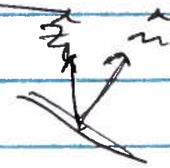
$c \to 0$, $b = a$ $\qquad S = 2\pi a$. $\qquad E = \frac{\pi}{2}$

$c \to a$, $b = 0$ $\qquad S = 4a$. $\qquad E \to 1$.

within surface: $(d\vec{x}) \cdot (\vec{\nabla}\phi) = 0$

$(d\vec{x}) \perp (\vec{\nabla}\phi)$.

$\boxed{\vec{\nabla}\phi \text{ is } \perp \text{ surface}}$.



$$\hat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$\cos\gamma = \hat{n}\cdot\hat{z} = \frac{(\vec{\nabla}\phi)\cdot\hat{z}}{|\vec{\nabla}\phi|} = \frac{\partial\phi/\partial z}{|\vec{\nabla}\phi|}.$$

$$\sec\gamma = \frac{|\vec{\nabla}\phi|}{|\vec{\nabla}\phi\cdot\hat{z}|} = \frac{\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]^{1/2}}{(\partial\phi/\partial z)}$$

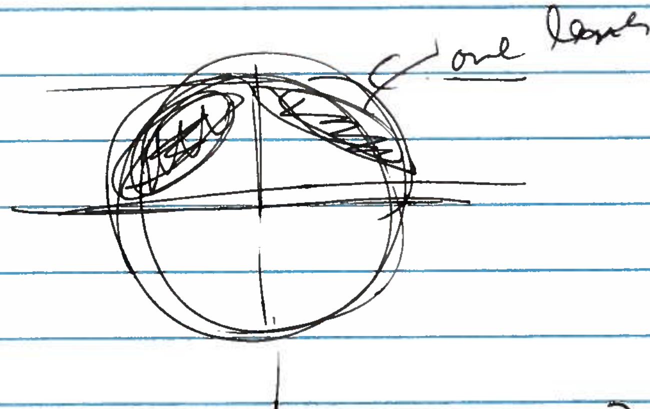
(A5) $z = f(x,y)$ $\qquad \phi = f(x,y) - z$

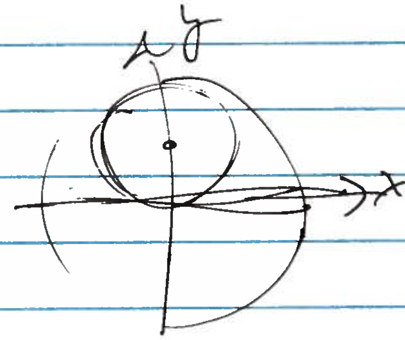
$$\frac{\partial\phi}{\partial z} = -1$$

$$\boxed{d^2A = dx\,dy\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}}$$

$$\boxed{d^2A = dx\,dy\,\frac{\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right)}{\left|\frac{\partial\phi}{\partial z}\right|}}$$ cf. (10.2cc)

Peculiar shape:
drill cylinder ($r = \frac{1}{2}$) through
top of sphere ($r = 1$)





Area of one (fingernail).
centered above ($x=0$, $y=\frac{1}{2}$)

$$x^2+\left(y-\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2 \qquad \boxed{x^2 = y - y^2}$$

Surface $\quad x^2+y^2+z^2-1 = \phi = 0 \qquad z(x,y)$.

$$\sec\gamma = \frac{\sqrt{(\phi_{,x})^2+(\phi_{,y})^2+(\phi_{,z})^2}}{|\phi_{,x}|} = \frac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{2z} = \frac{1}{z}$$

$$A = \int_S dx\,dy \cdot \sec\gamma = \int_0^1 dy \int_{-\sqrt{y-y^2}}^{+\sqrt{y-y^2}} dx \cdot \frac{1}{\sqrt{1-x^2-y^2}}$$

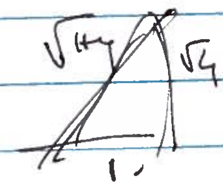
$$A = \int_0^1 dy \cdot 2\int_0^{\sqrt{y-y^2}} \frac{dx}{\sqrt{1-x^2-y^2}}$$

$x^2 = (1-y^2)\sin^2\theta$

$x = \sqrt{1-y^2}\,\sin\theta$

$$= \int_0^1 dy \cdot 2\int_0^{\theta} \frac{\cancel{\sqrt{1-y^2}}\,\cancel{\cos\theta}\, d\theta}{\sqrt{\cancel{(1-y^2)}(1-\cancel{\sin^2\theta})}} = 2\int_0^1 dy\, \sin^{-1}\left(\frac{\sqrt{y-y^2}}{\sqrt{1-y^2}}\right)$$

$$\sqrt{\frac{y(1-y)}{(1+y)(1-y)}} = \sin^{-1}\sqrt{\frac{y}{1+y}} = \tan^{-1}\sqrt{y}$$



$$= 2\int_0^1 dy\, \tan^{-1}\sqrt{y} = 2\left[(1+y)\tan^{-1}\sqrt{y} - \text{[illegible]}\right]_0^1$$

$$= 2\left(2\cdot\frac{\pi}{4} - 1\right) = \pi - 2$$

$$= 1.1415$$

$$\pi r^2 \cdot \sqrt{2} = \frac{\pi\sqrt{2}}{4} = \frac{(3.14)(1.41)}{4} = 1.11072$$