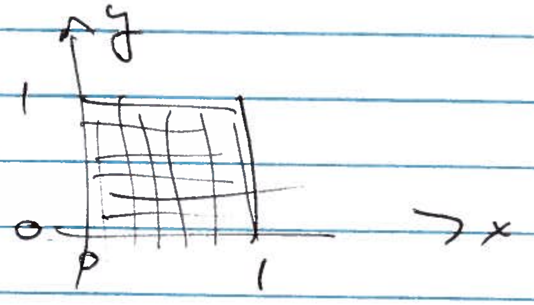
9/18/2017.

Chapter 6. Multiple integrals



$f(x,y) = x^2 + y^2 - 2xy$

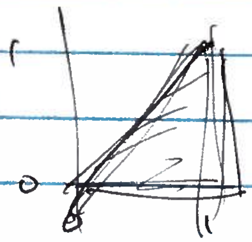
$$\int_0^1 dx \int_0^1 dy\, f(x,y) = \int_0^1 dx \left[x^2 y + \frac{1}{3} y^3 - 2x \cdot \frac{1}{2} y^2 \right)_{y=0}^{y=1}$$

$$= \int_0^1 dx \left(x^2 + \frac{1}{3} - x \right)$$

$$= \left(\frac{1}{3} x^3 + \frac{1}{3} x - \frac{1}{2} x^2 \right)_{x=0}^{x=1} = \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{2}{6} + \frac{2}{6} - \frac{3}{6} = \frac{1}{6}$$

$\int_0^1 dy \int_0^1 dx\, f(x,y)$. (symmetric)

Those aren't the hard ones.



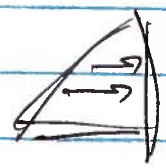
$$S = \left\{ x \geq 0,\ y \geq 0,\ \begin{matrix} x \geq y \\ y \leq x \end{matrix} \right\}$$

$$\to x - y \geq 0$$

$$\int_S dx\, dy\, f = \int_0^1 dx \int_0^x dy\, (x^2 + y^2 - 2xy)$$

$$= \int_0^1 dx \left(x^2 y + \frac{1}{3} y^3 - x \cdot y^2 \right) \Big|_{y=0}^{y=x}$$

$$= \int_0^1 dx \left(\cancel{x^3} + \frac{1}{3} x^3 - \cancel{x^3} \right) = \frac{1}{3} \cdot \frac{x^4}{4} \Big|_0^1 = \underline{\frac{1}{12}}$$



$$\int_S dx\, dy\, f = \int_0^1 dy \int_{1-y}^1 dx\, (x^2 + y^2 - 2xy)$$

$$= \int_0^1 dy \cdot \left(\frac{1}{3} x^3 + x y^2 - x^2 y \right) \Big|_{1-y}^{1}$$

$$= \int_0^1 dy \left(\frac{1}{3} + y^2 - y - \frac{1}{3}(1-y)^3 - (1-y) y^2 + (1-y)^2 y \right)$$

$$= \int_0^1 dy \Big[\cancel{\frac{1}{3}} - \frac{1}{3}(\cancel{1} - 3y + 3y^2 - y^3) + y^2 - (\cancel{y^2} - y^3)$$
$$- \cancel{y} + (\cancel{1} - 2y + y^2)\, y \Big]$$

$$= \int_0^1 dy \left[y - y^2 + \frac{1}{3} y^3 + y^3 - 2y^2 + y^3 \right]$$

$$\left(y - 3y^2 + \frac{7}{3} y^3 \right) = \frac{1}{2} - 1 + \frac{7}{12} = \frac{6 - 12 + 7}{12} = \frac{1}{12}$$

Changing variables $\quad y = y(x)$

$$\int_{y_0}^{y_1} dy\, f(y) = \int_{x_0}^{x_1} dx \cdot \frac{dy}{dx} \cdot f(y(x))$$

$\frac{dy}{dx} < 0$ reverses orientation, $x_1 < x_0$

$$\int_{x_0}^{x_1} dx\, \frac{dy}{dx} = \int_{x_1}^{x_0} dx \left(-\frac{dy}{dx}\right)$$

(reversed) (negative) $\quad$ (proper) (positive)

proper orientation $\quad \left|\frac{dy}{dx}\right|$

$x, y = (x(u,v), y(u,v))$.

$$\int dx\, dy\, f \longrightarrow \int du\, dv \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| f(x(u,v), y(u,v))$$

partial derivatives. (there are N^2 of them)

make one thing: determinant = "Jacobian"

Absolute value $\quad \left| \frac{\partial x}{\partial u}\frac{\partial v}{\partial y} - \frac{\partial x}{\partial v}\frac{\partial u}{\partial y} \right|$

Lots of applications.

2D. distribution of mass.

Area $= \int d^2S = \int_S dx\,dy$

Mass $\quad M = \int_S \rho\, d^2S.$

Center of mass $\quad \bar{x} = \dfrac{\int d^2S.\; x\rho}{M}.$

moments of inertia. $\quad I_{xx} = \int d^2S.\left(x^2 - \frac{1}{2}(x^2+y^2)\right)\rho.$

$\rho = \rho_0 \dfrac{x}{a}$

$A = \frac{1}{2}a^2 = \int dx\,dy$

$$M = \int_0^a dx \int_0^x dy.\; \rho_0 \frac{x}{a} = \int_0^a dx.\; \rho_0 \frac{x^2}{a} = \frac{\rho_0}{a}\cdot\frac{a^3}{3} = \frac{1}{3}\rho_0 a^2$$

$$M = \frac{1}{3}\rho_0 a^2 = \frac{2}{3}\rho_0 A$$

$$\bar{x} = \frac{\int_0^a dx \int_0^x dy.\; x\left(\rho_0 \frac{x}{a}\right)}{M} = \frac{\int_0^a dx.\; \frac{\rho_0}{a} x^3}{\frac{1}{3}\rho_0 a^2} = \frac{\frac{1}{4}\rho_0 a^4}{\frac{1}{3}\rho_0 a^3} = \frac{3a}{4}$$

$$\bar{y} = \frac{\int_0^a dx \int_0^x dy.\; \frac{\rho_0 x}{a}\cdot y}{M} = \frac{\int_0^a dx.\; \frac{\rho_0 x}{a}.\frac{x^2}{2}}{\frac{1}{3}\rho_0 a^2} = \frac{\frac{1}{8}\rho_0 a^4}{\frac{1}{3}\rho_0 a^3} = \frac{3a}{8}.$$

Polar:
$$x = r\cos\phi \qquad y = r\sin\phi$$
$$r = \sqrt{x^2+y^2} \qquad \tan\phi = y/x$$

$$\int dx\, dy\, f(x,y) = \int dr\, d\phi \left\| \frac{\partial(x,y)}{\partial(u,v)} \right\| f(r\cos\phi, r\sin\phi)$$

$$\frac{\partial x}{\partial r} = \cos\phi \qquad \frac{\partial x}{\partial \phi} = -r\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\phi \qquad \frac{\partial y}{\partial \phi} = r\cos\phi$$

$$J = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} = \left| r\cos^2\phi - (-r\sin^2\phi) \right|$$
$$= r(\cos^2\phi + \sin^2\phi) = r$$

$$\int dx\, dy\, f(x,y) = \int r\, dr\, d\phi\; f(r,\phi)$$

$f=1 \quad x^2+y^2 \le a^2$

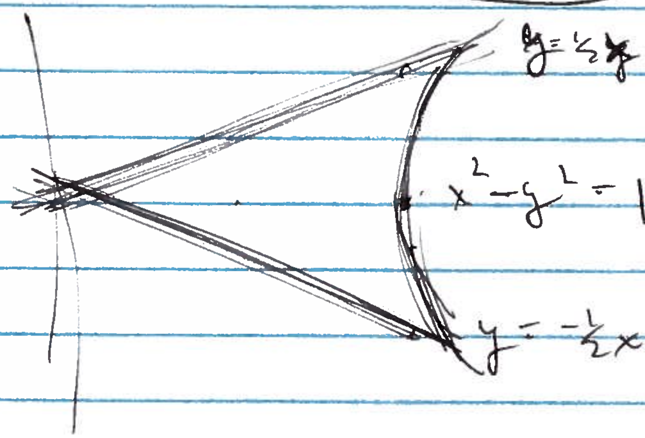
$$A = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy$$
$$= \int_0^a r\, dr \int_0^{2\pi} d\phi = \frac{1}{2}a^2 \cdot 2\pi = \pi a^2$$

(6)

$$I = \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2} = \int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}y^2} \quad \text{(before)}$$

$$I^2 = \int dx\,dy\, e^{-\frac{1}{2}(x^2+y^2)} = \int r\,dr\,d\theta\, e^{-\frac{1}{2}r^2} = 2\pi.$$

Hyperbolic coordinates



$x = \rho \cosh \xi$
$y = \rho \sinh \xi$

$\cosh \xi = \frac{1}{2}(e^{\xi} + e^{-\xi})$
$\sinh \xi = \frac{1}{2}(e^{\xi} - e^{-\xi})$

$\frac{d}{d\xi} \cosh \xi = \sinh \xi$

$\frac{d}{d\xi} \sinh \xi = \cosh \xi$

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \xi} \end{vmatrix} = \begin{vmatrix} \cosh \xi & \rho \sinh \xi \\ \sinh \xi & \rho \cosh \xi \end{vmatrix} = \rho(\cosh^2 \xi - \sinh^2 \xi) = \rho$$

7

$$A = \int_S dx\,dy = \int_S \rho\, d\rho\, d\varphi$$

Arrowhead. $\boxed{0 \le \rho < 1}$

$y = \pm \frac{1}{2} x \qquad \rho \sinh \varphi = \pm \frac{1}{2} \rho \cosh \varphi$

$$\boxed{\tanh \varphi = \pm \frac{1}{2}}$$

$$\frac{e^{\varphi} - e^{-\varphi}}{e^{\varphi} + e^{-\varphi}} = \frac{1}{2} \qquad e^{2\varphi} - 1 = \frac{1}{2}\left(e^{2\varphi} + 1\right)$$

$$\frac{1}{2} e^{2\varphi} = \frac{3}{2} \qquad e^{2\varphi} = 3 \qquad \boxed{\varphi_0 = \frac{1}{2} \ln 3}$$

$$A = \int_0^1 \rho\, d\rho \int_{-\varphi_0}^{\varphi_0} d\varphi = \left(\frac{1}{2}\right)(2\varphi_0)$$

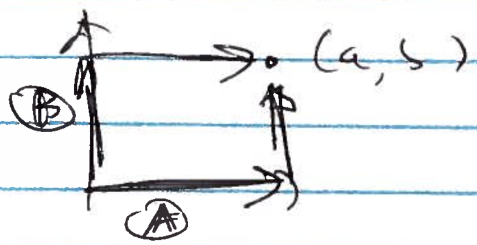
$$= \varphi_0 = \frac{1}{2} \ln 3 = \ln \sqrt{3} = 0.549306\ldots$$

curves, surfaces;

$$\int_a^b \left[f(x,y)\,dx + g(x,y)\,dy\right] = ??$$

Not always true that $f\,dx + g\,dy = dF$

Value can depend on path.



(A) follow $(y=0) \to (x=a)$

(B) follow $(x=0) \to (y=b)$

(1) $y\,dx - x\,dy$

(2) $x\,dx + y\,dy$

(1). (A). $$\int_A (y\,dx - x\,dy) = \int_{0\,(y=0)}^{a} dx\,(\cancel{y}\,dx - x\,\cancel{dy}) + \int_{0\,(x=a)}^{b} dy\,(y\,\cancel{dx} - x\,dy)$$

$$= -ab.$$

(B) $$\int_B (y\,dx - x\,dy) = \int_{0\,(x=0)}^{b} dy\,(y\,\cancel{dx} - \cancel{x}\,dy) + \int_{0\,(y=b)}^{a} (y\,dx - x\,\cancel{dy}) = +ab$$

(2) $\int_A (x\,dx - y\,dy) = \int_0^a (x\,dx - y\,dy)\Big|_{y=0} - \int_0^b (x\,dx - y\,dy)\Big|_{x=a}$

$= \int_0^a x\,dx - \int_0^b y\,dy = \frac{1}{2}a^2 - \frac{1}{2}b^2$

$\int_B (x\,dx - y\,dy) = \int_0^b (x\,dx - y\,dy)\Big|_{x=0} + \int_0^a (x\,dx - y\,dy)\Big|_{y=b}$

$= -\int_0^b y\,dy + \int_0^a x\,dx = -\frac{1}{2}b^2 + \frac{1}{2}a^2$

How do you know when path independent?

$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$

$= f(x,y)\,dx + g(x,y)\,dy.$

$\frac{\partial^2 F}{\partial y\,\partial x} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial^2 F}{\partial x\,\partial y} = \frac{\partial}{\partial x} g(x,y)$

$\left(\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\right)$ "exact differential"

else, "inexact" $\neq dF$.