9/15/17 "Implicit differentiation"

$x + e^{x} = t$ $\quad \frac{dx}{dt} = ??$

$dx + e^{x} dx = dt$ $\quad \frac{dx}{dt} = \frac{1}{1+e^{x}}$

In general: x-y 2d plane

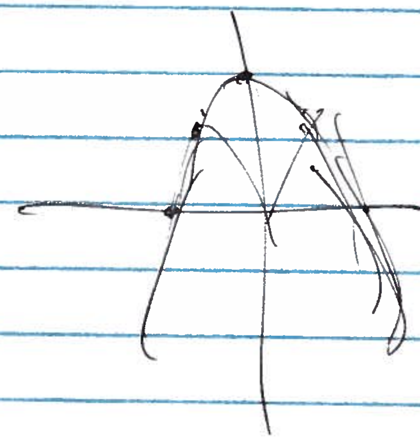
$f(x,y) = 0$ 1-d curve.

$df = \left(\frac{\partial f}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y}\right) dy = 0$

$$\boxed{\frac{dx}{dy} = - \frac{\partial f/\partial x}{\partial f/\partial y}} = \left(\frac{\partial x}{\partial y}\right)_f$$

$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$ revisited

Closest point to origin on curve $y = 1-x^2$



(I). $d^2 = x^2 + y^2 = x^2 + (1-x^2)^2$

$= x^4 - x^2 + 1.$

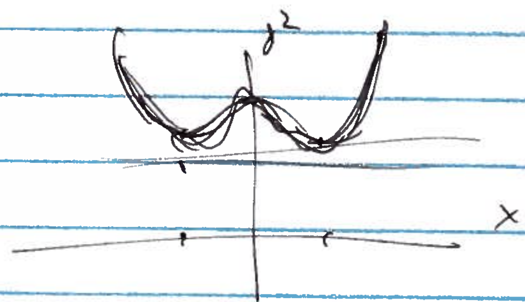
$\frac{d d^2}{dx} = 4x^3 - 2x = 0 \qquad 2x(2x^2-1) = 0$

$x=0 \qquad 2x^2 - 1 = 0$

$x = \frac{1}{\sqrt{2}}$

$y = 1 \qquad y = \frac{1}{2}$

$d^2 = 0 + 1^2 = 1 \qquad d^2 = x^2 + y^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4}$




§ 5.9 Constraints

"Lagrange multiplier"

minimize $f(x,y)$

subject to $\phi(x,y) = 0$

let $\boxed{F = f + \lambda\phi}$

minimize $F \rightarrow \frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial \lambda} = 0$

$$F = x^2 + y^2 + \lambda(x^2 + y - 1)$$

$$\frac{\partial f}{\partial x} = 2x + 2\lambda x = 2x(1+\lambda)$$

$$\frac{\partial f}{\partial y} = 2y + \lambda$$

$$\frac{\partial f}{\partial \lambda} = x^2 + y - 1$$

$x = 0 \rightarrow y = 1 \rightarrow \lambda = -2 \qquad d^2 = 0 + 1 = 1$

$\lambda = -1 \rightarrow y = \frac{1}{2} \rightarrow x^2 - \frac{1}{2} = 1 \qquad x^2 = \frac{1}{2} \qquad x = \pm\frac{1}{\sqrt{2}}$

$d^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ ✓

§5.12: differentiation of integrals

$$\boxed{F(x,t) = \int dt\, f(x,t)}$$

$$\Longleftrightarrow \quad \frac{\partial F}{\partial t} = f(x,t)$$

$$\frac{\partial^2 F}{\partial t\, \partial x} = \frac{\partial^2 F}{\partial x\, \partial t}$$

$$\rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial f}{\partial x}$$

$$\int dt \rightarrow \boxed{\frac{\partial F}{\partial x} = \int dt \left(\frac{\partial f}{\partial x}\right)} \quad (5.46)$$

Leibnitz: order of integration, differentiation can be reversed (uniform convergence)

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$$I(x) = \int_{t=u(x)}^{t=v(x)} dt\, f(x,t)$$

$$= F(x, v(x)) - F(x, u(x))$$

$$\frac{\partial I}{\partial v} = f(x, v(x)) \qquad \frac{\partial I}{\partial u} = -f(x, u(x))$$

(as before)

$$\frac{dI}{dx} = \frac{\partial I}{\partial x} + \frac{\partial I}{\partial v}\frac{dv}{dx} + \frac{\partial I}{\partial u}\frac{du}{dx}$$

$$\boxed{\frac{dI}{dx} = \int_u^v \frac{\partial f}{\partial x}\, dt + f(x,v)\frac{dv}{dx} - f(x,u)\frac{du}{dx}}$$

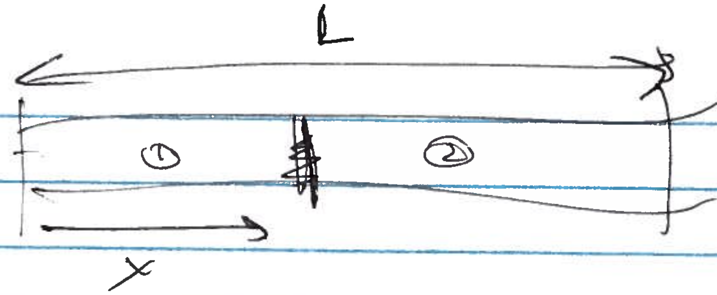
$$I(x) = \int_x^{x^2} \frac{\sin xt}{t}\, dt$$

$$\frac{dI}{dx} = \int_x^{x^2} \frac{\cancel{t}\cos xt}{\cancel{t}}\, dt + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \left(\frac{1}{x}\sin xt\right)_x^{x^2} + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \frac{\sin x^3}{x} - \frac{\sin x^2}{x} + \frac{2\sin x^3}{x} - \frac{\sin x^2}{x}$$

$$= \frac{3\sin x^3 - 2\sin x^2}{x}$$




$$F = +p_1 A - p_2 A$$

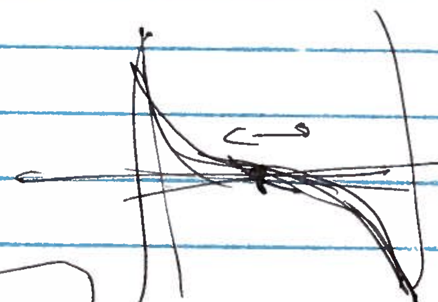
$$p_1 V_1 = p_1 \cdot A \cdot x = N_1 kT = p_0 A x_0$$

$$p_2 V_2 = p_2 A (L - x) = N_2 kT = p_0 A (L - x_0)$$

$p_1 = p_2 = p_0$ @ $x = x_0$ $\qquad \frac{p_0 A x_0}{N_1} = kT = \frac{p_0 A (L - x_0)}{N_2}$

$$N_2 x_0 = N_1 (L - x_0)$$

$$F = \frac{A p_0 x_0}{x} - \frac{A p_0 (L - x_0)}{L - x}$$



$$m \frac{d^2 x}{dt^2} = p_0 A \left( \frac{x_0}{x} - \frac{L - x_0}{L - x} \right)$$

$x = x_0 + \Delta x$ $\qquad \cancel{\frac{x_0}{x_0 (1 + \frac{\Delta x}{x_0})} - \frac{(L - x_0)}{(L - x_0)}}$

$$\frac{x_0}{x} - \frac{L - x_0}{L - x} = -\frac{L}{x_0 (L - x_0)} (\Delta x)$$

$$\omega^2 = \frac{k}{m} = \frac{p_0 A L}{x_0 (L - x_0)}$$

$$m \frac{d^2 \Delta x}{dt^2} = -p_0 A \cdot \frac{L}{x_0 (L - x_0)} \Delta x$$

$$k_{eff} = \frac{p_0 A L}{x_0 (L - x_0)}$$

$$\frac{\text{Force}}{\text{Area}} \cdot \frac{L^3}{L^2} = \frac{\text{Force}}{\text{Length}}$$