8/30/2017

Alternating series

$S_{2m} > 0$ $S_{2m} < a_1$

(Bounded, monotonic $\to$ converges).

$$S = \left(\sum_{n=0}^{N} a_n\right) + \left(\sum_{N+1}^{\infty} a_n\right)$$

S_N remainder

Same argument $\to$ $r_N > 0$ $r_N < a_{N+1}$

"Error is less than ~~first~~ next omitted term"

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \cdots$$

$$= \sum_n \binom{p}{n} x^n = \sum_n \frac{p!}{n!(p-n)!} x^n$$

$$\left| \frac{\frac{p!}{(n+1)!(p-n-1)!} x^{n+1}}{\frac{p!}{n!(p-n)!} x^n} \right| = \left| \frac{n!}{(n+1)!} \frac{(p-n)!}{(p-n-1)!} x \right|$$

$$= \left| \left(\frac{p-n}{n+1}\right) x \right| \to |x|.$$

(2)

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ is <u>unique</u>.

$f(x) = \sum a_n x^n = g(x) = \sum b_n x^n$ $\{a_n\} = \{b_n\}$

$f(0) = a_0 = g(0) = b_0$.

$f'(0) = a_1 = g'(0) = b_1$ etc

multiplication ~ polynomials, term by term

$(a_0 + a_1 x + a_2 x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots)$

$= a_0 + a_0 b_1 x + b_0 a_1 x$

$+ (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$

$$f(x) = 1 + \frac{3}{x} + \frac{3(1+x)^{1/2}}{x^{3/2}} \ln\left[(1+x)^{1/2} - x^{1/2}\right]$$

$(x \to \infty)$ $\quad 1 + \frac{3}{x} + \frac{3}{x} \ln\left[(x^{1/2})(1 + \frac{1}{2x}) - x^{1/2}\right]$

$$= 1 + \frac{3}{x} + \frac{3}{x}\ln\left(\frac{1}{2}\frac{1}{x^{1/2}}\right) \to 1 + O\left(\frac{\ln x}{x}\right)$$

$(x \to 0)$ $\quad \frac{1}{x^{3/2}}$?, $\quad x^{-3/2}, x^{-1}, (x^{-1/2}), 1, x^{1/2}, x$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}x^3$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

$$\log\left((1+x)^{1/2} - x^{1/2}\right) = \log\left(1 - x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)$$

$O(x^{5/2})$

$$\simeq \left(-x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2\right) - \frac{1}{2}\left(-x^{1/2} + \frac{1}{2}x\right)^2 + \frac{1}{3}\left(-x^{1/2}\right)^3$$

$+O(x^3)$ $\quad -\frac{1}{8}x^2$

$$= -x^{1/2} + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{2}\left(x - 2\cdot\frac{1}{2}x^{1/2}\cdot x + \frac{1}{4}x^2\right) - \frac{x^{3/2}}{3}$$

$$= -x^{1/2} + \left(\frac{1}{2} - \frac{1}{3}\right)x^{3/2} + O(x^{5/2})$$

$$-\frac{3}{40}x^{5/2}$$

(4)

$$(1+x)^{1/2}\log\left((1+x)^{1/2}-x^{1/2}\right)$$

$$\simeq\left(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2\right)\left(-x^{1/2}+\tfrac{1}{6}x^{3/2}-\tfrac{3}{40}x^{5/2}\right)$$

$$=\left(-x^{1/2}\right)+\left(\tfrac{1}{2}x\left(-x^{1/2}\right)+\tfrac{1}{6}x^{3/2}\right)$$

$$+\left(+\tfrac{1}{8}x^{5/2}+\tfrac{1}{12}x^{5/2}-\tfrac{3}{40}x^{5/2}\right)$$

$$=-x^{1/2}-\tfrac{1}{3}x^{3/2}+\tfrac{2}{15}x^{5/2}+\cdots$$

$$1+\frac{3}{x}+\frac{3(1+x)^{1/2}}{x^{3/2}}\log\left((1+x)^{1/2}-x^{1/2}\right)$$

$$\simeq 1+\frac{3}{x}+\frac{3\left(-x^{1/2}-\tfrac{1}{3}x^{3/2}+\tfrac{2}{15}x^{5/2}+\cdots\right)}{x^{3/2}}$$

$$\simeq\tfrac{2}{5}x-\tfrac{8}{35}x^2+\cdots$$

(5)

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

$$\sinh\left[\ln\left(\sqrt{1+x} - \sqrt{x}\right)\right]$$

$$= \frac{1}{2}\left(\left(\sqrt{1+x} - \sqrt{x}\right) - \frac{1}{\sqrt{1+x} - \sqrt{x}} \cdot \frac{\sqrt{1+x} + \sqrt{x}}{\sqrt{1+x} + \sqrt{x}}\right)$$

$$= \frac{1}{2}\left(\sqrt{1+x} - \sqrt{x} - \left(\frac{\sqrt{1+x} + \sqrt{x}}{(1+x) - x}\right)\right) = -\sqrt{x}$$

$$f = 1 + \frac{3}{x} + \frac{3\sqrt{1+x}}{x^{3/2}}\left(-\sinh^{-1}\sqrt{x}\right)$$

(only odd powers of $\sqrt{x}$)

→ integer powers of x.

Physics – Blackbody radiation

Volume density of radiation (energy) per λ emitted by perfect BB @ T:

$$u(\lambda)\,d\lambda = \frac{8\pi hc}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1}\,d\lambda.$$

$$x = \frac{\lambda kT}{hc}$$

$[kT]$ = energy = J
$[h]$ = J·s
$[hc]$ = J·m

x = dimensionless

$$[u] = \frac{J\cdot m}{m^5} = \frac{(J/m^3)}{m} = \frac{\text{energy density}}{\text{(wave)-length}}$$

$$u = 8\pi hc \cdot \left(\frac{kT}{hc}\right)^5 \left(\frac{hc}{\lambda kT}\right)^5 \cdot \frac{1}{e^{hc/\lambda kT}-1}$$

$$= 8\pi \cdot \frac{(kT)^5}{(hc)^4}\,\frac{1}{x^5}\,\frac{1}{e^{1/x}-1}$$

$x \to 0$. $\begin{matrix} x \to 0 \\ T \to 0 \end{matrix}$ $\frac{1}{x} \to$ big $e^{1/x} \to$ BIG $\frac{1}{x^5} \frac{1}{e^{1/x}}$?

$\log = -5 \log x - \frac{1}{x} = -\frac{1}{x}(1 + 5x\log x)$

$x \to 0$ $x \log x = \frac{\log x}{1/x} \to \frac{(1/x)}{(-1/x^2)} \to -x \to 0$

$\frac{1}{x^5} \frac{1}{e^{1/x}} \to 0$.

$x \to \infty$. $e^{1/x} \to 1 + \frac{1}{x} + \frac{1}{2}\frac{1}{x^2} + \ldots$

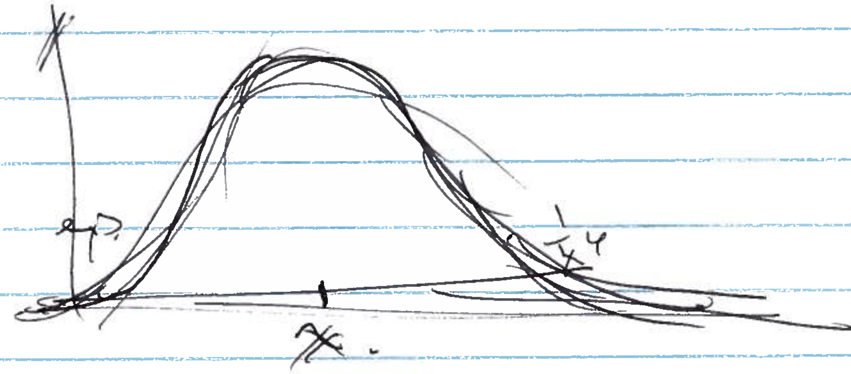
$e^{1/x} - 1 \to (1 + \frac{1}{x} + \ldots) - 1 \to \frac{1}{x}$.

$x^5 (e^{1/x} - 1) \to x^4 + \frac{1}{2}x^3 + \frac{1}{6}x^2 + \ldots$

$\to$ big.

$\frac{1}{x^5} \frac{1}{e^{1/x} - 1} \to 0$.

$\frac{1}{x^5} \frac{1}{e^{1/x}-1}$



(peak) . $\frac{d}{dx}\left(\frac{1}{x^5}\frac{1}{e^{1/x}-1}\right) = -\frac{5x\, e^{1/x} - 5x - e^{1/x}}{x^7 \left(e^{1/x}-1\right)^2}$

vanishes at $x_0 = 0.201405$

x small, $e^{1/x}$ big $\quad (5x-1)e^{1/x} = 0$

$x \simeq \frac{1}{5}$

$e^5 = 148.413$

$\lambda_0 = 0.201 \frac{hc}{kT}$

$T_\odot = 5500K$. $\quad kT \simeq 0.474$ eV $= 7.59 \times 10^{-20}$ J.

$\lambda_0 = 526$ nm $= 5260$ Å Yellow-green.