8/28/17. Sums of series $\sum a_n$.

need: $a_n \to 0$ (preliminary)

Other tests:

Comparison. $|a_n| < m_n$ "known"

$|a_n| \geq d_n$.

"absolute convergence"

Integral. $\sum a_n \leftrightarrow \int dx\, a(x)$.

Alternating, strictly alternating

monotonic

$|a_n| \to 0$.

Ratio test $\left|\frac{a_{n+1}}{a_n}\right| < 1$.

(comparison with geometric $\sum r^n$)

"Ratio comparison" ("special")

Σb_n . converges $\frac{a_n}{b_n} \to$ (finite constant)

$\Longleftrightarrow \Sigma a_n$ converges.

Σd_n diverges $\frac{a_n}{d_n} \to$ (finite constant or ∞.)

$\Longleftarrow \Sigma a_n$. diverges.

Next step $\Sigma a_n \longrightarrow \Sigma a_n x^n$

power series

Some known:

$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$

$\frac{x^{n+1}}{x^n} = x$ converges $|x| < 1$

pole @ $x \to 1$

③

$$\int_0^x \frac{dx'}{1-x'} = -\ln(1-x)$$

$$= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

Converges $|x| < 1$.

$x \to 1$ $\quad \log(1-x)$ diverges "logarithmically".

$x \to -1$ $\quad \ln 2$ ok $\quad$ alternating harmonic

$$x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$$

$$= x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{n=1}^{\infty} n x^n$$

$$\frac{(n+1)x^{n+1}}{n x^n} \to \left(1 + \frac{1}{n}\right)x = x$$ $\quad$ Converges $|x| < 1$

$x \to 1$. $\quad \frac{1}{(1-x)^2}$ $\quad$ pole of order 2

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Mr. Taylor's Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

evens $\rightarrow$ $\cosh x = \frac{1}{2}(e^x + e^{-x})$

odds $\rightarrow$ $\sinh x = \frac{1}{2}(e^x - e^{-x})$

signs $\rightarrow$ $\sin x$, $\cos x$ $\quad e^{ix} \pm e^{-ix}$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \binom{p}{n} x^n \qquad \frac{p!}{n!(p-n)!}$$

even if $p \neq$ integer.

$$\frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1} \rightarrow 0 \quad \text{for all } x$$

$(n \to \infty)$

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Bessel function:

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k+m}.$$

$$\left| \frac{\dfrac{(-1)^{k+1}}{(k+1)!(k+m+1)!} \left(\frac{x}{2}\right)^{2(k+1)\cancel{+m}}}{\dfrac{(-1)^k}{k!(k+m)!} \left(\frac{x}{2}\right)^{2k\cancel{+m}}} \right|$$

$$= \left| (-1) \frac{1}{(k+1)} \frac{1}{(k+m+1)} \left(\frac{x}{2}\right)^2 \right|$$

$$= \frac{x^2/4}{(k+1)(k+m+1)} \to 0 \quad (k \to \infty)$$

for any x

Example (p. 440) (revised).

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)}$$ $\sim \frac{1}{n^2}$

$$\left| \frac{\frac{(-1)^{n+2} x^{n+1}}{(\cancel{n+1})(n+2)}}{\frac{(-1)^{n+1} x^n}{n(\cancel{n+1})}} \right| = \left| \frac{-x \cdot n}{(n+2)} \right|$$

$$= |x| \left| \frac{n}{n+2} \right| < 1 \qquad |x| < 1,$$

$x = -1$: $$\sum_{n=1}^{\infty} \frac{(-)}{n(n+1)} = - \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= -\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} \cdots\right)$$
$$+ \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots\right) = -1.$$

$x = 1$: $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} = \sum (-1)^n \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$
$$- \left(\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} - \cdots\right)$$

$$= 2 \sum \frac{(-1)^{n+1}}{n} - 1 = 2 \ln 2 - 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\int_0^x \frac{dx'}{1+x'} = \ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Again

$$\int_0^x dx' \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x'^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \frac{x^{n+1}}{n+1}$$

$$= x \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n(n+1)}.$$

$$= \int_0^x dx' \ln(1+x')$$

$$= x\ln(1+x) + \ln(1+x) - x$$

$$(1+x)\ln(1+x) - x$$

$$f(x) = \frac{(1+x)\ln(1+x) - x}{x}$$

$x \to 1$ $\quad \frac{2\ln 2 - 1}{1}$ $\quad$ $x \to -1$ $\quad \frac{(0) - (-1)}{(-1)}$

$= -1$

Manipulations

(*) Can add, subtract term-by-term
converges in common interval

(*) Can multiply, divide (if $d \neq 0$). $\frac{\sin x}{x}$
If both have a convergent interval
result will have some interval of convergence

(*) Can compound: $f(g(x))$ $\sum a_n x_n$, $\sum b_n x^n$

$$\Rightarrow \sum_{n=0}^{\infty} a_n \left(\sum_{k=0}^{\infty} b_k x^k \right)^n$$

$$= a_0 + a_1 b_0 + a_2 b_0^2 + a_3 b_0^3 + \dots$$

$$+ (a_1 + 2a_2 b_0 b_1 + 3a_3 b_0^2 b_1 + \dots) x$$

$b_0 = 0$ $\quad a_0 + a_1 b_1 x + (a_2 b_1^2 + a_1 b_2) x^2 + \dots$

finite polynomials $O(x^p)$ $\rightarrow$ algebra $O(x^p)$