8/25/17 Sequences + Series — Ch. 4

a_n. $n = (0), 1, \ldots$

$$\frac{n^2 + 5n^3}{2n^3 + 3\sqrt{4 + n^6}} = \frac{n^3(5 + 1/n)}{n^3\left(2 + 3\sqrt{1 + 4/n^6}\right)} \to 1.$$

$a_n = \left(\frac{1}{n}\right)^{1/n}$.

n	
1	1
10	0.794
100	0.955
10^3	0.9931
10^6	0.99935

$\log a_n = \frac{1}{n} \ln \frac{1}{n} = -\frac{\ln n}{n}$

L'Hôpital $\to -\frac{(1/n)}{1} \to 0$

p. 142-143

$$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} \to 1$$

$\sum_{n=1}^{\infty} a_n$. $S_N = \sum_{n=1}^{N} a_n$ $S = \lim_{N\to\infty} S_N$

converges $\Longleftrightarrow$ $S_N \to$ [illegible]

(2)

$\boxed{\text{Harmonic series}}$ (split)

$$\sum_{n=1}^{N} \frac{1}{n} \to ?, \quad \text{(diverges?)}$$

$$\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \to ?? \quad \text{(converges?)}$$

How do you know?. ($a_n \to 0$) (test)

= "Can you prove it"?
(practical, not formal.)

First hint - not (?)

grouping

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + 8 \times \frac{1}{16} + \cdots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

→ (diverges)

(3)

Special case "alternating series"

4.3.3

$u_n \geq 0$ $\quad$ $u_{n+1} < u_n$, $u_n < u_{n-1}$ decreasing ($n \geq N$)

$a_n = (-1)^{n+1} u_n$.

$\sum a_n = \sum (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots$

$N = 1$

$S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m})$

all terms positive. $\quad$ $S_{2m} \geq 0$ increasing.

$S_{2m} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m}$

all positive

$S_{2m} \leq u_1$ $\quad$ S_{2m} strictly increasing

$\Rightarrow$ converges

Theorem 3.14 Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Rudin Principles of Mathematical Analysis

(4)

$\sum_{n=1}^{N} \frac{1}{n} \sim \ln N$

$\left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right) \longrightarrow \gamma_E = 0.5772156649\ldots$

$\sum_{n=1}^{} (-1)^{n+1} \frac{1}{n} \Longrightarrow$

$N = 99$	0.698172
$N = 100$	0.688172

many $\to$ 0.693147... $= \underline{\ln 2}$.

will see in <u>power series</u>.

Comparison test

Known, somehow, $m_n \geq 0$ $\sum_{n=1}^{\infty} m_n$ converges

$|a_n| \leq m_n$ $\quad m_n \to 0$

squeezed.

$\Rightarrow \sum a_n$ converges (better than)

$\sum d_n$ diverges,

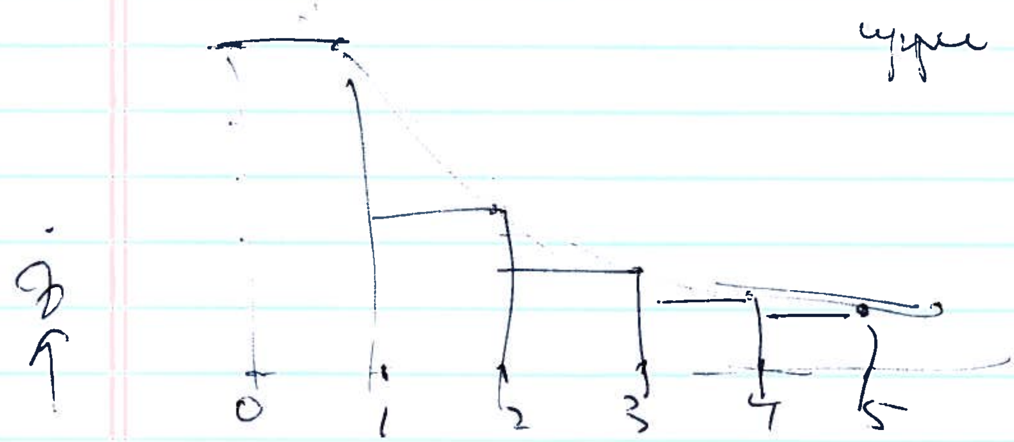
$|a_n| \geq d_n \to \sum a_n$ diverges

(worse than)

Harmonic

Compare sum, integral.

upper bound.



$\ln N + \ln\left(1+\frac{1}{N}\right) < \sum_{n=1}^{N} \frac{1}{n} < 1 + \ln N$

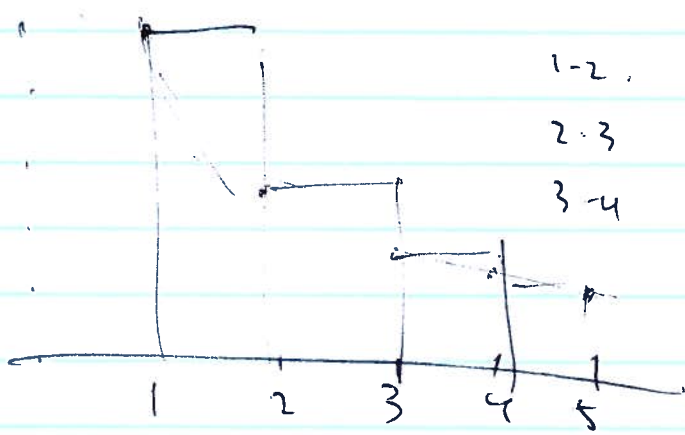
0-1	$1 < \frac{1}{x}$
1-2	$\frac{1}{2} < \frac{1}{x}$
2-3	$\frac{1}{3} < \frac{1}{x}$
3-4	$\frac{1}{4} < \frac{1}{x}$

$$\sum_{n=1}^{N} \frac{1}{n} < \int_0^N \frac{dx}{x}$$

bad choice

$$\sum_{n=1}^{N} \frac{1}{n} < 1 + \int_1^N \frac{dx}{x}$$

$$= 1 + \ln N$$



1-2,	$1 > x$
2-3	$\frac{1}{2} > x$
3-4	$\frac{1}{3} > x$

$$\sum_{n=1}^{N} \frac{1}{n} > \int_1^{N+1} \frac{dx}{x} = \ln(N+1)$$

$$= \underline{\ln N + \ln\left(1+\frac{1}{N}\right)}$$

Generalize: Integral Test

$a_n \geq 0$ positive ($a_n \geq a_{n+1}$) monotonic

(perhaps only for large enough n. — Finitely many terms don't affect convergence)

$\sum_n a_n$ converges if $\int^\infty dx\, a(x)$ converges,

$\sum_n a_n$ diverges if $\int^\infty dx\, a(x)$ diverges

$(N \to \infty)$

$$\ln N + \ln\left(1 + \frac{1}{N}\right) < \sum_{n=1}^{N} \frac{1}{n} < 1 + \ln N.$$

$$\boxed{0 < \left(\sum_{n=1}^{N} \frac{1}{n} - \ln N\right) < 1}$$

$$\sum_{n=1}^{N} \frac{1}{n} - \ln N \to \gamma_E$$

A series that converges for $|r| < 1$.

$a_n = r^n$ $\qquad S_N = \sum_{n=0}^{N} r^n$

$= 1 + r + \cdots + r^N$

$$(1-r)S_N = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^N}$$
$$\qquad - \cancel{r} - \cancel{r^2} - \cdots - \cancel{r^N} - r^{N+1}$$

$$S_N = \frac{1-r^{N+1}}{1-r}$$

$N \to \infty$ $\quad S_N \to \dfrac{1}{1-r}$