8/23/17 Gaussian integral

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

§ 6.4.2

Γ-function | § 18.12.1

B-function | § 18.12.2

Stirling's approximation | (§ 18.159)

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$$\Gamma(x) = \int_0^\infty t^{x-1}\, dt\; e^{-t}$$

$$\Gamma(x+1) = \int_0^\infty \underbrace{t^x}_{u}\, \underbrace{dt\; e^{-t}}_{dv}$$

$$= [uv]_0^\infty - \int_0^\infty v \cdot du \qquad x > 0$$

$$= -\int_0^\infty (-e^{-t})(x\, t^{x-1}\, dt)$$

$$= x \cdot \int_0^\infty t^{x-1}\, dt\; e^{-t}$$

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

$\Gamma(1) = 1$ $\qquad \Gamma(2) = 1\,\Gamma(1) = 1$

$\Gamma(3) = 2 \cdot \Gamma(2) = 2$

$\Gamma(4) = 3\,\Gamma(3) = 6$

$\Gamma(n+1) = n!$ $\qquad n! = n\,(n-1)!$

(3)

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty t^{\frac{1}{2}-1}\, dt\, e^{-t}$$ ($t = x^2/2$, $dt = x\,dx$)

$$= \int_0^\infty \left(\tfrac{1}{2}x^2\right)^{-\frac{1}{2}} (x\,dx)\, e^{-\frac{1}{2}x^2}$$

$$= \sqrt{2} \int_0^\infty dx\, e^{-\frac{1}{2}x^2} = \sqrt{2} \cdot \tfrac{1}{2} \cdot \sqrt{2\pi}$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

n	$n!$	$n^n e^{-n} \sqrt{2\pi n}$	ratio
1	1	0.922	
2	2	1.919	0.9595
3	6	5.836	0.9727
10	3628800		0.99170
100			0.999167
1000			0.9999166

$$\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots\right)$$

"Asymptotic"

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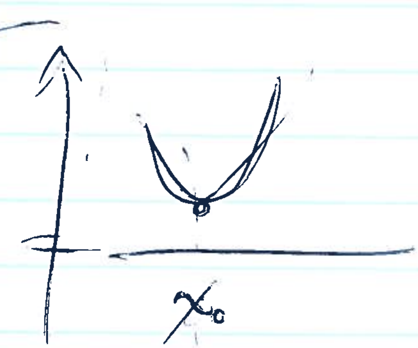
$$n! = \int x^n dx\, e^{-x} = \int dx\, e^{-(x - n \ln x)}$$

$$I = \int dx\, e^{-f(x)}$$

f has minimum @ x_0



$$f(x) \simeq f(x_0) + f'_0 (x - x_0) + f''_0 \tfrac{1}{2} (x - x_0)^2 \ldots$$

minimum

$$I \simeq \int dx\, e^{-[f_0 + \frac{1}{2} f''_0 (x-x_0)^2]}$$

$$= e^{-f_0} \int dx\, e^{-\frac{1}{2}\frac{(x-x_0)^2}{1/f''}}$$

$$= e^{-f_0} \sqrt{2\pi} \sqrt{\tfrac{1}{f''_0}}$$

$f = x - n \ln x$

$f' = 1 - \frac{n}{x}$ $\quad$ $x_0 = n$

$f'' = +\frac{n}{x^2}$ $\quad$ $\sqrt{\frac{1}{f''}} = \frac{1}{\sqrt{n}}$

$$n! \simeq e^{-(n - n \ln n)} \sqrt{\tfrac{2\pi}{n}} = n^n e^{-n} \sqrt{2\pi n}$$

Beta-function

$$B(p,q) = \int_0^1 dt\; t^{p-1}(1-t)^{q-1} = B(q,p)$$

$t = \sin^2\theta$

$dt = 2\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\frac{\pi}{2}} d\theta\; \sin^{2p-1}\theta \cdot \cos^{2q-1}\theta$$

$$\Gamma(p)\Gamma(q) = \int_0^\infty dt\; t^{p-1} e^{-t} \int_0^\infty du\; u^{q-1} e^{-u}$$

$$= \int x\,dx \left(\tfrac{1}{2}x^2\right)^{p-1} e^{-\frac{1}{2}x^2} \cdot \int y\,dy \left(\tfrac{1}{2}y^2\right)^{q-1} e^{-\frac{1}{2}y^2}$$

$$= \int dx\,dy \left(\tfrac{1}{2}\right)^{p+q-2} x^{2p-1} y^{2q-1} e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \int r\,dr\,d\theta \left(\tfrac{1}{2}\right)^{-1}\left(\tfrac{1}{2}\right)^{p+q-1} r^{2p+2q-2} e^{-\frac{1}{2}r^2}$$

$$\times (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}$$

$$= 2\int ds\; s^{p+q-1} e^{-s} \cdot \int d\theta\, (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}$$

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$