**PHZ 3113** Fall 2017

Homework #9, Due Monday, December 4

1. Let the differential operator $\mathcal{D}f = \dfrac{d}{dx}\,(1-x^2)\,\dfrac{df}{dx}$ act on functions defined on the interval $(-1,1)$.

$(a)$ Show that for any functions $f(x)$, $g(x)$, $\displaystyle\int_{-1}^{1} dx\, f(x)\,[\mathcal{D}g(x)] = \int_{-1}^{1} dx\,[\mathcal{D}f(x)]\,g(x)$; that is, that $\mathcal{D}$ is self-adjoint.

$(b)$ Suppose $P_m$ and $P_n$ are eigenfunctions of $\mathcal{D}$, such that $\mathcal{D}P_m(x) + m(m+1)\,P_m(x) = 0$ and $\mathcal{D}P_n(x) + n(n+1)\,P_n(x) = 0$. Show that for $n \neq m$, $P_m(x)$ and $P_n(x)$ are orthogonal on $(-1,1)$.

$(c)$ Verify that $P_0 = 1$ and $P_1 = x$ are eigenfunctions with the suggested eigenvalues.

$(d)$ Let $P_2(x)$ be a polynomial of degree 2, $P_2 = c_0 + c_1\,x + c_2\,x^2$. Choose $c_0$ and $c_1$ so that $P_2$ is orthogonal to $P_0$ and $P_1$. Choose $c_2$ so that $P_2(1) = 1$. Write your resulting $P_2(x)$. Is your $P_2$ an eigenfunction of $\mathcal{D}P_2 + p(p+1)\,P_2$ for some eigenvalue $p$?

2. Let $f(x)$ be the triangular function that takes on the value $f = 0$ for $|x| > a$, joined linearly from the points at $x = \pm a$ to $f = 1/a$ at $x = 0$ (see plot on next page).

$(a)$ Compute the Fourier transform $\tilde{f}(k)$.

$(b)$ What does $\tilde{f}(k)$ become as $a \to 0$? What does $f(x)$ become as $a \to 0$?

$(c)$ Compute the value of the integral $I_1 = \displaystyle\int_{-\infty}^{\infty} \frac{(1-\cos y)}{y^2}\,dy$.

$(d)$ Compute the value of the integral $I_2 = \displaystyle\int_{-\infty}^{\infty} \frac{(1-\cos y)^2}{y^4}\,dy$.