PHZ 3113 Fall 2017

Homework #8, Due Monday, November 20

1. *Let* $f(x) = \begin{cases} \sin x & 0 < x < \pi \ , \\ 0 & -\pi < x < 0 \ . \end{cases}$

(*a*) *Compute the coefficients* a_n, b_n *to express* $f(x)$ *as a Fourier series. (Be careful with* $n = 1$.*)*

For $n \neq 1$, the cosine coefficients are

$$a_n = \frac{1}{\pi}\int_0^\pi \sin x \, \cos nx \, dx = -\frac{1+\cos n\pi}{\pi(n^2-1)} = \begin{cases} -\dfrac{2/\pi}{n^2-1} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}$$

(this also holds for a_0), and the sine coefficients are

$$b_n = \frac{1}{\pi}\int_0^\pi \sin x \, \sin nx \, dx = \frac{\sin n\pi}{n^2-1} = 0.$$

For $n = 1$,

$$a_1 = \frac{1}{\pi}\int_0^\pi \sin x \, \cos x \, dx = 0, \qquad b_1 = \frac{1}{\pi}\int_0^\pi \sin^2 x \, dx = \frac{1}{2}.$$

Thus,

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k)^2-1}. \qquad \blacksquare$$

(*b*) *Compute the value of the sum* $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)^2-1}$.

Evaluate $f(x)$ at $x = \frac{\pi}{2}$:

$$f(\frac{\pi}{2}) = 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\cos k\pi}{(2k)^2-1}.$$

So,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)^2-1} = \frac{\pi}{2}\left(1 - \frac{1}{2} - \frac{1}{\pi}\right) = \frac{\pi}{4} - \frac{1}{2}. \qquad \blacksquare$$

(*c*) *Compute the value of the sum* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{[(2k)^2-1]^2}$.

The square of coefficients appears in Parseval's theorem,

$$\int_0^{2\pi} f^2(x) = \int_0^{\pi} \sin^2 x\, dx = \frac{\pi}{2} = \frac{\pi}{2}\, a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

and so

$$\frac{1}{2}\left(\frac{2}{\pi}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{[(2k)^2-1]^2} = \frac{1}{2}.$$

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{[(2k)^2-1]^2} = \frac{\pi^2}{4}\left(\frac{1}{2} - \frac{1}{4} - \frac{2}{\pi^2}\right) = \frac{\pi^2}{16} - \frac{1}{2}. \qquad \blacksquare$$

2. *Because sine and cosine can be written as complex exponentials and vice versa, a Fourier series on the interval* $(-\pi, \pi)$ *can just as well be written as*

$$f(x) = \sum_{m=-\infty}^{\infty} c_m \, e^{imx}.$$

(a) *Examine the integral* $\int_0^{2\pi} dx \, e^{-ipx} \, f(x)$ *and write an expression for* c_p.

Invert the order of sum and integral,

$$\int_0^{2\pi} dx \, e^{-ipx} \, f(x) = \int_0^{2\pi} dx \sum_{m=-\infty}^{\infty} C_m \, e^{i(m-p)x} = \sum_{m=-\infty}^{\infty} C_m \int_0^{2\pi} dx \, e^{i(m-p)x}.$$

For $p \neq m$, since $m - p$ is a (nonzero) integer, the integral vanishes

$$\int_0^{2\pi} dx \, e^{i(m-p)x} = \left[\frac{e^{i(m-p)x}}{i(m-p)}\right]_{x=0}^{x=2\pi} = \frac{1}{i(m-p)} \left[e^{2\pi i(m-p)} - 1\right] = 0,$$

while for $m = p$ the integral is just $\int dx \, (1) = 2\pi$, and

$$\sum_{m=-\infty}^{\infty} C_m \int_0^{2\pi} dx \, e^{i(m-p)x} = \sum_{m=-\infty}^{\infty} C_m \, 2\pi \, \delta_{mp} = 2\pi \, C_p.$$

Thus,

$$C_p = \frac{1}{2\pi} = \int_0^{2\pi} dx \, e^{-ipx} \, f(x). \qquad \blacksquare$$

(b) *Compute the Fourier coefficients* c_m *for the Dirac* $\delta(x)$ *(plus periodic repetitions).*

From (a),

$$c_m = \frac{1}{2\pi} \int dx \, e^{-imx} \, \delta(x) = \frac{1}{2\pi} \, e^{-imx}\Big|_{x=0} = \frac{1}{2\pi}. \qquad \blacksquare$$

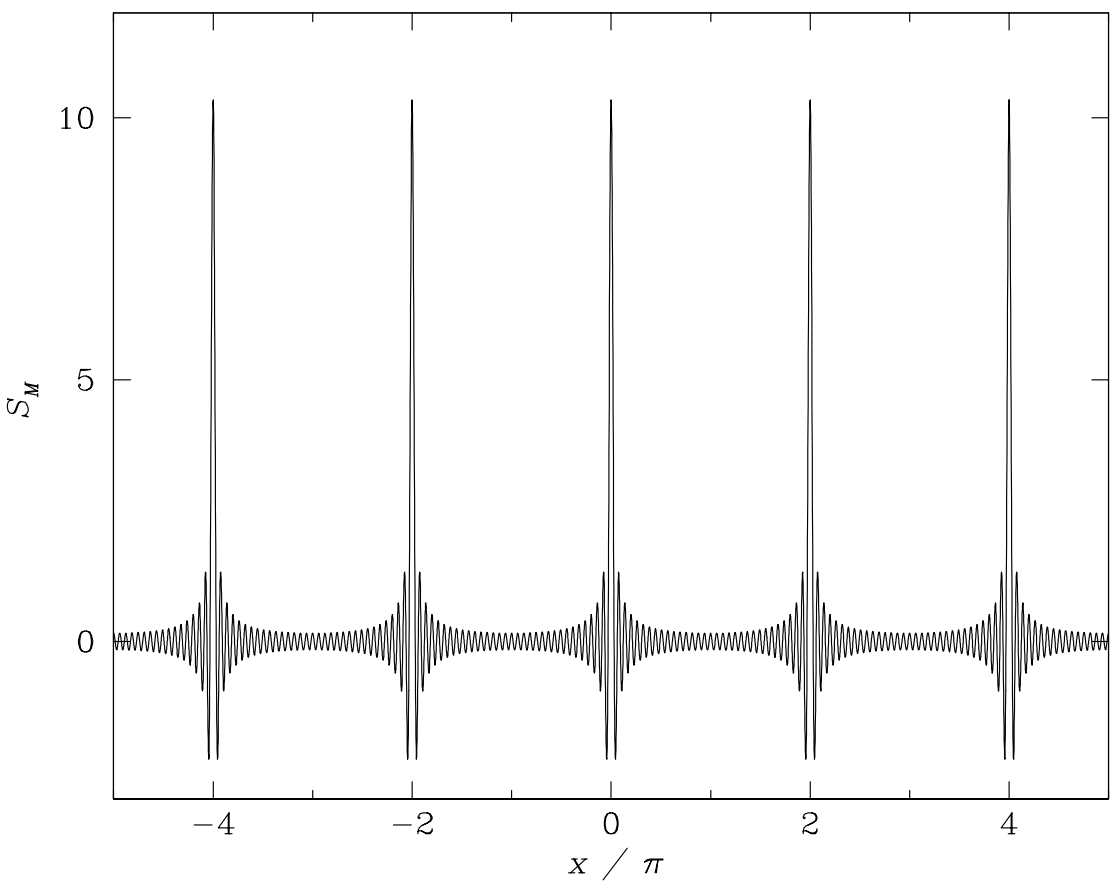
Yes, that's it.

(*c*) *Compute the partial sum* δ_M *of the series including terms up to order* M*. Sketch a plot for some moderate value of* M*.*

$$\sum_{m=-M}^{M} \frac{1}{2\pi}\, e^{imx} = \frac{e^{-iMx}}{2\pi} \sum_{m=0}^{2M} (e^{ix})^m = \frac{e^{-iMx}}{2\pi}\, \frac{1 - e^{i(2M+1)x}}{1 - e^{ix}}$$

$$= \frac{1}{2\pi}\, \frac{e^{i(M+1)x} - e^{-iMx}}{e^{ix} - 1} = \frac{1}{2\pi}\, \frac{e^{i(M+\frac{1}{2})x} - e^{-i(M+\frac{1}{2})x}}{e^{i(\frac{1}{2}x)} - e^{-i(\frac{1}{2}x)}} = \frac{\sin[(M + \frac{1}{2})x]}{2\pi\, \sin(\frac{1}{2}\, x)}. \blacksquare$$

The plot shows this function for $M = 32$.



(Note that this problem is different from the example considered in class.)