PHZ 3113 Fall 2017

Homework #4, Due Wednesday, October 4

1. *Define elliptical coordinates u, v by* $x = \sqrt{u^2 + c^2}\,\cos v, \quad y = u\,\sin v\,.$

(a) *Write the area element $dA = dx\,dy$ in terms of elliptical u, v.*

The element of area is $dx\,dy = |J|\,du\,dv$, where the Jacobian determinant is

$$J = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \dfrac{u\cos v}{\sqrt{u^2+c^2}} & -\sqrt{u^2+c^2}\sin v \\ \sin v & u\,\cos v \end{pmatrix}$$

$$= \frac{u^2}{\sqrt{u^2+c^2}}\,\cos^2 v + \sqrt{u^2+c^2}\,\sin^2 v = \frac{u^2 + c^2\sin^2 v}{\sqrt{u^2+c^2}}. \qquad \blacksquare$$

(b) *Using elliptical coordinates, compute the area of an ellipse with semimajor axis a, semiminor axis b, where $a^2 = b^2 + c^2$.*

It is easy to see that curves of constant u are ellipses,

$$\frac{x^2}{u^2+c^2} + \frac{y^2}{u^2} = \cos^2 v + \sin^2 v = 1,$$

and $0 < u < b$ and $0 < v < 2\pi$ traces the full ellipse. The area is

$$A = \int_0^b du \int_0^{2\pi} dv\,\frac{u^2 + c^2\sin^2 v}{\sqrt{u^2+c^2}} = \pi\int_0^b du\,\frac{2u^2+c^2}{\sqrt{u^2+c^2}}.$$

Since

$$\frac{d}{du}\left(u\sqrt{u^2+c^2}\right) = \frac{2u^2+c^2}{\sqrt{u^2+c^2}},$$

this is

$$A = \left[\pi u\sqrt{u^2+c^2}\right]_0^b = \pi b\sqrt{b^2+c^2} = \pi ab. \qquad \blacksquare$$

2. *Let the scalar field f be given by* $f(x,y) = \dfrac{x^2 - y^2}{\sqrt{x^2+y^2}}$.

(a) *Compute the components of the gradient ∇f. Where is the derivative troublesome?*

The derivatives are

$$\frac{\partial f}{\partial x} = \frac{x^3 + 3xy^2}{(x^2+y^2)^{3/2}}, \qquad \frac{\partial f}{\partial y} = -\frac{3x^2y + y^3}{(x^2+y^2)^{3/2}},$$

and

$$\nabla f = \frac{(x^3 + 3xy^2)\,\hat{\boldsymbol{x}} - (3x^2y + y^3)\,\hat{\boldsymbol{y}}}{(x^2+y^2)^{3/2}}. \qquad \blacksquare$$

This is well-behaved except as $x^2 + y^2 \to 0$.

(b) *Compute the directional derivative of $\nabla_{\boldsymbol{u}} f$ in the direction $\boldsymbol{u} = \hat{\boldsymbol{x}}\,\cos\alpha + \hat{\boldsymbol{y}}\sin\alpha$. Write your answer in terms of polar coordinates ρ, ϕ, where $x = \rho\cos\phi$, $y = \rho\sin\phi$. What is the directional derivative along a radial ray (fixed ϕ) as $\rho \to 0$?*

The directional derivative is

$$\frac{df}{du} = \boldsymbol{u}\cdot\nabla f = \frac{\cos\alpha\,(x^3 + 3xy^2) - \sin\alpha\,(3x^2y + y^3)}{(x^2+y^2)^{3/2}}. \qquad \blacksquare$$

In terms of ρ and ϕ, this is

$$\begin{aligned}\frac{df}{du} &= \cos\alpha\,(\cos^3\phi + 3\cos\phi\sin^2\phi) - \sin\alpha\,(3\cos^2\phi\sin\phi + \sin^3\phi) \qquad \blacksquare \\ &= \cos\alpha\,(4\cos^3\phi + 3\cos\phi) - \sin\alpha\,(3\sin\phi - 2\sin^3\phi). \qquad \blacksquare\end{aligned}$$

Either of these is correct, but using $\cos 3\phi = 4\cos^3\phi - 3\cos\phi$, $\sin 3\phi = 3\sin\phi - 4\sin^3\phi$, this can also be written

$$\frac{df}{du} = \tfrac{1}{2}\left[\cos\alpha\,(3\cos\phi - \cos 3\phi) - \sin\alpha\,(3\sin\phi + \sin 3\phi)\right] = \tfrac{3}{2}\,\cos(\phi+\alpha) - \tfrac{1}{2}\,\cos(3\phi - \alpha). \quad \blacksquare$$

This is perhaps easier to obtain by writing

$$f = \rho\,\cos 2\phi, \qquad \nabla f = \frac{\partial f}{\partial\rho}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}}, \qquad \frac{df}{du} = \cos 2\phi\;\hat{\boldsymbol{\rho}}\cdot\boldsymbol{u} - 2\sin 2\phi\;\hat{\boldsymbol{\phi}}\cdot\boldsymbol{u}.$$

with

$$\hat{\boldsymbol{\rho}}\cdot\boldsymbol{u} = \cos\phi\cos\alpha + \sin\phi\sin\alpha = \cos(\alpha-\phi) \qquad \hat{\boldsymbol{\phi}}\cdot\boldsymbol{u} = -\sin\phi\cos\alpha + \cos\phi\sin\alpha = \sin(\alpha-\phi),$$

leading to linear combinations of $\cos[2\phi \pm (\alpha - \phi)]$, as above.

3. *Let the vector field* $\boldsymbol{v}$ *be*

$$\boldsymbol{v} = 2xy\,\hat{\boldsymbol{x}} + (x^2 + y^2)\,\hat{\boldsymbol{y}}\,.$$

(*a*) *Compute* $\nabla \cdot \boldsymbol{v}$. *Compute* $\nabla \times \boldsymbol{v}$.

The divergence is

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = (2y) + (2y) = 4y \qquad \blacksquare$$

The curl is

$$\nabla \times \boldsymbol{v} = \left(\hat{\boldsymbol{x}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial}{\partial y}\right) \times (\hat{\boldsymbol{x}}\,v_x + \hat{\boldsymbol{y}}\,v_y) = \hat{\boldsymbol{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = [(2x) - (2x)]\,\hat{\boldsymbol{z}} = 0 \qquad \blacksquare$$

(*b*) *Compute* $\nabla(\nabla \cdot \boldsymbol{v})$ *and* $\nabla^2\boldsymbol{v}$. *Compute* $\nabla \times (\nabla \times \boldsymbol{v})$ *two different ways.*

The gradient of the divergence is

$$\nabla(\nabla \cdot \boldsymbol{v}) = \left(\hat{\boldsymbol{x}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial}{\partial y}\right)(4y) = 4\,\hat{\boldsymbol{y}} \qquad \blacksquare$$

The Laplacian is

$$\nabla^2\boldsymbol{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[2xy\,\hat{\boldsymbol{x}} + (x^2 + y^2)\,\hat{\boldsymbol{y}}\right] = (2 + 2)\,\hat{\boldsymbol{y}} = 4\,\hat{\boldsymbol{y}} \qquad \blacksquare$$

Since $\nabla \times \boldsymbol{v} = 0$, we have $\nabla \times (\nabla \times \boldsymbol{v}) = 0$, also the result of the other way,

$$\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla^2\boldsymbol{v} = (4\,\hat{\boldsymbol{y}}) - (4\,\hat{\boldsymbol{y}}) = 0 \qquad \blacksquare$$

4. *The electric and gravitational forces between point particles are both inverse square, and in analogy with electrostatics, the gravitational field* $\boldsymbol{g}$ *can be obtained from a gravitational potential* $\boldsymbol{g} = -\nabla\Phi$ *that satisfies* $\nabla^2\Phi = 4\pi G\rho$. *In numerical simulations of large-scale cosmological structure, it is sometimes useful to "soften" the Newtonian gravitational potential* $\Phi = -GM/r$ *of a point particle of mass* M *as*

$$\Phi_a = -\frac{GM}{\sqrt{r^2 + a^2}}.$$

(*a*) *What is the gravitational field* $\boldsymbol{g}$ *derived from* Φ_a*? How does* $\boldsymbol{g}$ *behave as* $r \to \infty$*? Show that the softening introduces a maximum in the gravitational force between particles.*

$$\boldsymbol{g} = -\nabla\Phi_a = -\frac{GMr}{(r^2 + a^2)^{3/2}}\,\hat{\boldsymbol{r}}. \qquad \blacksquare$$

This behaves as GM/r^2 as $r \to \infty$, vanishes linearly as $r \to 0$, and has peak magnitude $(2/3\sqrt{3})\,GM/a^2$ at $r = a/\sqrt{2}$.

(*b*) *The softened* Φ_a *can be interpreted as the potential of a slightly fuzzy particle with a mass density* $\rho_a(r)$*. What is* $\rho_a(r)$*? What is the total mass,* $M_a = \int d^3v\,\rho_a$*? What radius* $r_{1/2}$ *contains half the mass?*

From Poisson's equation,

$$\nabla^2\Phi_a = \frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} = \frac{3a^2GM}{(r^2 + a^2)^{5/2}} = 4\pi G\rho, \qquad \rho_a(r) = \frac{3a^2M}{4\pi(r^2 + a^2)^{5/2}}. \qquad \blacksquare$$

The mass within radius r is

$$M_a(r) = \int_0^r 4\pi r^2\,\rho_a(r) = \frac{Mr^3}{(r^2 + a^2)^3},$$

and so $M(r \to \infty) = M$, independent of a. The half-mass radius is

$$\frac{r_{1/2}^2}{r_{1/2}^2 + a^2} = \frac{1}{2^{2/3}}, \qquad r_{1/2} = \frac{a}{\sqrt{2^{2/3} - 1}} = 1.30477\,a. \qquad \blacksquare$$

As $a \to 0$, this is a mass distribution that has zero density for $r \neq 0$, has infinite density as $r \to 0$, has extent of order a, and has constant integral total mass. Thus, as $a \to 0$ this returns to the point mass, or

$$\rho_a \to M\,\delta^{(3)}(\boldsymbol{r}).$$