PHZ 3113 Fall 2017

Homework #1, Due Friday, September 1

1. *Let* $a_n = \left(1 + \dfrac{\alpha}{n}\right)^n$. *What is* $\lim_{n\to 0} a_n$? *What is* $\lim_{n\to\infty} a_n$?

In many cases like this, it is convenient to look at the logarithm: if the log has a limit, then so does the original sequence, and the limit of the log is the log of the limit.

For small n, using the series $\ln x = x + \frac{1}{2}\,x + \frac{1}{3}\,x^2 + \cdots$ (§4.6.3)

$$\ln a_n = n\ \ln\!\left(1 + \frac{\alpha}{n}\right) = n\left[\ln\!\left(\frac{\alpha}{n}\right) + \ln\!\left(1 + \frac{n}{\alpha}\right)\right] = n\left[\ln\alpha - \ln n + \frac{n}{\alpha} + \mathcal{O}(n^2)\right].$$

The limit of $n\ln n$ is found from l'Hôpital's rule,

$$\lim_{n\to 0}\ n\ln n = \lim_{n\to 0}\frac{\ln n}{1/n} = \lim_{n\to 0}\frac{1/n}{(-1/n^2)} = \lim_{n\to 0}\,(-n) \to 0,$$

and

$$\lim_{n\to 0}\ln a_n \to 0.$$

For large n, again using the series for $\ln x$,

$$\ln a_n = n\ \ln\!\left(1 + \frac{\alpha}{n}\right) = n\left(\frac{\alpha}{n} - \frac{1}{2}\,\frac{\alpha^2}{n^2} + \cdots\right) = \alpha - \frac{1}{2}\,\frac{\alpha^2}{n} + \cdots \to \alpha.$$

Thus,

$$\lim_{n\to 0}\left(1 + \frac{\alpha}{n}\right)^n = 1, \qquad \lim_{n\to\infty}\left(1 + \frac{\alpha}{n}\right)^n = e^{\alpha}.$$

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2. *Decide whether the following sums converge or diverge. (Provide reasons for your choices.)*

$(a)\quad \displaystyle\sum_{n=2}^{\infty} \frac{1}{n\,\ln n \ln(\ln n)}$

The derivative of $\ln(\ln(\ln x))$ is

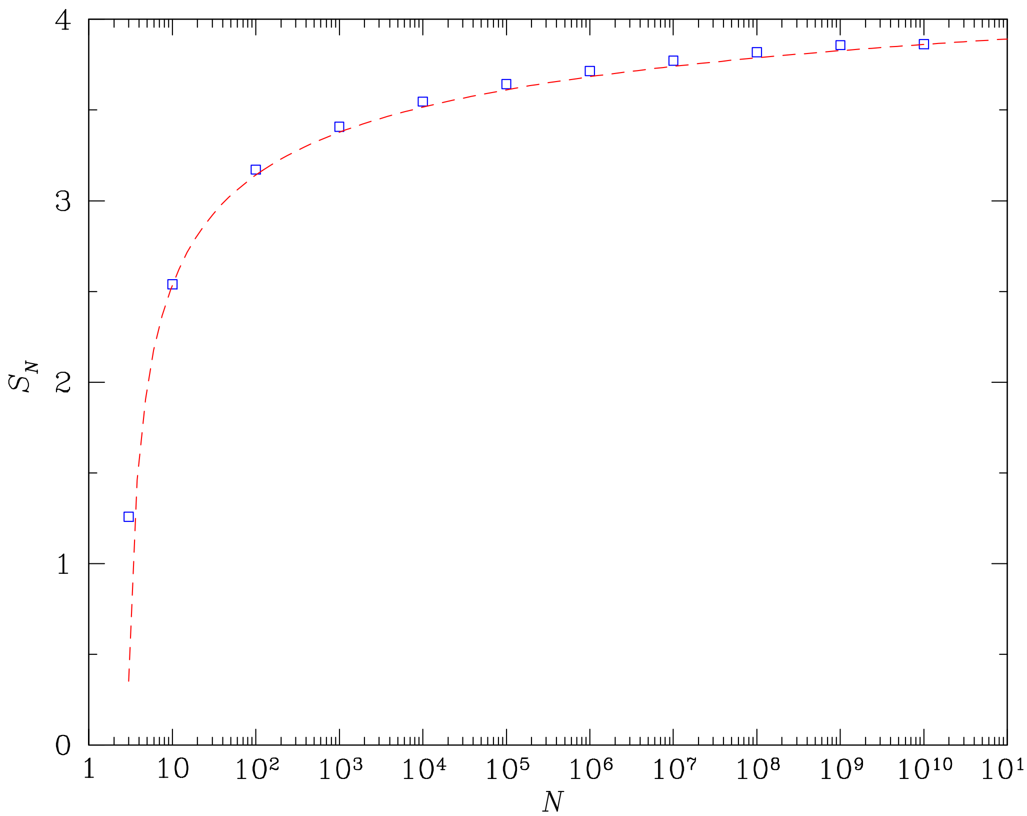
$$\frac{d}{dx}\ln(\ln(\ln x)) = \frac{1}{\ln(\ln x)}\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln(\ln x)}\frac{1}{\ln x}\frac{d}{dx}\ln x = \frac{1}{\ln(\ln x)\,\ln(x)\,x},$$

which means that running this chain in the other direction we know how to integrate the last expression. Thus, by the integral test, this sum diverges,

$$\sum_{n=2}^{N} \frac{1}{n\,\ln n\,\ln(\ln n)} \sim \int^{N} \frac{dx}{x\,\ln x\,\ln(\ln x)} = \ln(\ln(\ln N)) \to \infty. \qquad \blacksquare$$

The divergence is very slow: The sum of 10 terms is 2.53945; the sum of 100 terms is 3.11987; 1000 terms is 3.35467; 10^6 terms is 3.66112; 10^9 terms is 3.85829.

The plot shows partial sums (blue squares) and the function $\ln(\ln(\ln n))$ (red dashed line).



$$(b) \quad \sum_{n=1}^{\infty} \frac{\sin n}{n}$$

Since $|\sin n| \le 1$, this sum is bounded by $|a_n| \le b_n = 1/n$; but, $\sum b_n$ does not converge (the harmonic series again). The factor $\sin n$ introduces fluctuating signs, and so we can hope for some cancellation; but signs do not alternate systematically. But the integral converges,

$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2},$$

and that is the best justification that the series converges. Writing $\sin n$ as a sum,

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} n^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \sum_{n=1}^{\infty} n^{2k}$$

does not lead anywhere useful that I can see. (Although it does lead *Mathematica* to conclude that the value of the sum is $-\frac{1}{2}$, since the sum on n is $\zeta(-2k)$ [ζ is the Riemann zeta function]; and $\zeta(0) = -\frac{1}{2}$, $\zeta(-2k) = 0$ for integer $k > 0$. The failure is exchanging the order of sums, which is not justified here.)

This sum in fact converges to the value

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} = \operatorname{Im}\left[-\ln(1 - e^{i})\right] = \frac{\pi - 1}{2}.$$

$$(c) \quad \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

Since

$$\frac{1}{n} - \frac{1}{n+2} = \frac{2}{n(n+2)} \sim \frac{2}{n^2},$$

the sum converges by the integral test. Or, write

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots,$$

and notice that all terms from $1/3$ on cancel; the sum is $1 + \frac{1}{2} = \frac{3}{2}$.

3. *Consider the following alternating series*

$$\frac{2}{1} - \frac{1}{2} + \frac{2}{3} - \frac{1}{4} + \frac{2}{5} - \frac{1}{6} + \frac{2}{7} - \frac{1}{8} + \frac{2}{9} - \frac{1}{10} + \cdots .$$

Does the general term go to zero? Does the "Test for alternating series" in Section 4.3.3 apply? Does the series converge?

The general term is $2/n$ for odd n or $-1/n$ for even n, and so terms alternate in sign and go to zero. But, they do not go to zero monotonically, and so the premises of the test for alternating series are not satisfied. In the order written, the partial sums approach $\frac{3}{2}\ln N + 1.212397$ (empirical observation).

The series can be written as the alternating harmonic series plus the odd harmonic series, and so diverges to $+\infty$, or as twice the alternating series less the even harmonic series, and so diverges to $-\infty$ (note that these constitute two different orderings), or to any value in between.

4. *For what values of* p *does the sum* $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$ *converge? In terms of* $\zeta(p)$*, what is the sum over only even integers? Only odd integers? What is* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$*?*

By the integral test, for large N

$$\sum^{N} \frac{1}{n^p} \sim \int^{N} \frac{dx}{x^p} \sim \frac{1}{N^{p-1}}.$$

This converges for $p > 1$ and diverges for $p < 1$. For $p = 1$, the we have once again the harmonic series, which we know diverges.

The other variations do not introduce anything new: a factor of 2^p can be taken out of the sum over even integers,

$$\sum_{\text{even } n} \frac{1}{n^p} = \sum_{k=1}^{\infty} \frac{1}{(2k)^p} = \frac{1}{2^p} \sum_{k=1}^{\infty} \frac{1}{k^p} = \frac{1}{2^p} \zeta(p).$$

The sum over odd integers is the sum over all integers less the sum over even integers,

$$\sum_{\text{odd } n} \frac{1}{n^p} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^p} = \sum_{\text{all } n} \frac{1}{n^p} - \sum_{\text{even } n} \frac{1}{n^p} = \left(1 - \frac{1}{2^p}\right) \zeta(p),$$

and the alternating sum is

$$\sum \frac{(-1)^{n+1}}{n^p} = \sum_{\text{odd}} \frac{1}{n^p} - \sum_{\text{even}} \frac{1}{n^p} = \left(1 - \frac{1}{2^p}\right) \zeta(p) - \frac{1}{2^p} \zeta(p) = \left(1 - \frac{1}{2^{p-1}}\right) \zeta(p).$$

5. *The* partition function *for a quantum harmonic oscillator with energy levels* $\epsilon_n = (n+\frac{1}{2})\,\hbar\omega$ *is* $Z = \sum_{n=0}^{\infty} e^{-\beta\epsilon_n} = \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\beta\hbar\omega}$. *Compute* Z.

This is just the geometric series. The sum is

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})\beta\hbar\omega} = e^{-\frac{1}{2}\beta\hbar\omega} \sum_{n=0}^{\infty} [e^{-\beta\hbar\omega}]^n \\ &= e^{-\frac{1}{2}\beta\hbar\omega} \left[1 + e^{-\beta\hbar\omega} + (e^{-\beta\hbar\omega})^2 + (e^{-\beta\hbar\omega})^3 + \cdots\right] \\ &= \frac{e^{-\frac{1}{2}\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{e^{\frac{1}{2}\,\beta\hbar\omega} - e^{-\frac{1}{2}\,\beta\hbar\omega}} = \frac{1}{2}\,\text{csch}\,\frac{1}{2}\,\beta\hbar\omega. \end{aligned}$$

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6. *Take the alternating harmonic series* $\sum(-1)^{n+1}/n$ *and sum it in the following order: take the first two two (positive) odd terms and the first (negative) even term; then the next two odd terms and the next even term, two more odd terms and one even term, and so on. Does the sum in this ordering converge? If so, what is the resulting value of the sum? Since infinite precision, even if achievable, is not practical, make some choices and include an estimate of the accuracy of your result.*

As a start, partial sums of triplets are

$$1+\frac{1}{3}-\frac{1}{2}=\frac{5}{6}=0.833333$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}=\frac{389}{420}=0.926190$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}=\frac{13327}{13860}=0.961544$$

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8}=\frac{353201}{360360}=0.980134$$

Recalling the $1/n\ln n\ln\ln n$ series, this does not yet prove anything. But, note that the series can be rendered

$$\sum_{k=0}^{\infty}\left(\frac{1}{4k+1}+\frac{1}{4k+3}-\frac{1}{2k+2}\right)=\sum_{k=0}^{\infty}\frac{8k+5}{2(k+1)(4k+1)(4k+3)};$$

for large k the general term, call it a_n, goes to $b_n=1/4k^2$, which converges, say, by the integral test; and the series $\sum a_n$ thus converges by ratio comparison with b_n.

After summing N triplets, the remainder can be estimated as the integral

$$\int_N^{\infty}dx\left(\frac{1}{4x+1}+\frac{1}{4x+3}-\frac{1}{2x+2}\right)=\frac{1}{4}\log\left[\frac{4^2(N+1)^2}{(4N+1)(4N+3)}\right]$$
$$=\frac{1}{4}\log\left[\frac{(1+1/N)^2}{(1+1/4N)(1+3/4N)}\right]\approx\frac{1}{4N}.$$

Thus, even the sum for $N=4$, as above, is expected to be accurate to 6%. The sum of a million terms gives the value 1.039720758. The exact value is $\ln\sqrt{8}=1.03970$.