GR Notes

Lecture by Bernard Whiting

Notes by July Thomas

9/26/18

1 Wrapping up geodesic deviation

Last lecture, we showed the non-Euclidian space parameterized by s and t with vectors

$$S^\mu = \frac{dx^\mu}{ds} \tag{1}$$
$$T^\mu = \frac{dx^\mu}{dt} \tag{2}$$

$\star$If we adapt the coordinates to the geodesic parameters, then $S_\mu = (0, 0, , 1, 0)$ (where the 1 is in the s direction) and $T_\mu = (1, 0, 0, , 0)$ (where the 1 is in t).

The distance one travels in s changes as they move up and down (i.e. "in t").

Velocity of s along t

$$V^\mu = \nabla_T S^\mu \tag{3}$$
$$= T^\nu \nabla_\nu S^\mu \tag{4}$$

Acceleration

$$A^\mu = T^\sigma \nabla_\sigma (T^\nu \nabla_\nu S^\mu) \tag{5}$$

By condition $\star$ the commutator vanishes:

$$[S,T] = 0 \to S^\mu \nabla_\mu T^\nu = T^\mu \nabla_\mu S^\nu \tag{6}$$

Apply this when distributing ∇_σ in (5) to get

$$\begin{aligned}
&= T^\sigma \nabla_\sigma (S^\nu) \nabla_\nu T^\mu + T^\sigma S^\nu \nabla_\sigma (\nabla_\nu T^\mu) && (7)\\
&= S^\sigma \nabla_\sigma T^\nu \nabla_\nu T^\mu + T^\sigma S^\nu (\nabla_\nu \nabla_\sigma T^\mu + R^\mu_{\rho\sigma\nu} T^\rho) && (8)\\
&= R^\mu{}_{\rho\sigma\nu} T^\rho T^\sigma S^\nu && (9)
\end{aligned}$$

where we have used the fact that $T^\sigma S^\nu \nabla_\nu (\nabla_\sigma T^\mu) = S^\nu \nabla_\nu (T^\sigma \nabla_\sigma T^\mu) - S^\nu (\nabla_\nu T^\sigma)(\nabla_\sigma T^\mu)$.

Revisiting the 2D sphere Recall from several days ago that

$$R^\theta{}_{\phi\theta\phi} = \sin^2\theta. \tag{10}$$

Lowering the index gives (for a fixed sphere)

$$R_{\theta\phi\theta\phi} = a^2 \sin^2\theta. \tag{11}$$

Switching the indices gives

$$R_{\phi\theta\theta\phi} = -a^2 \sin^2\theta. \tag{12}$$

Finally, raise the first ϕ index to get

$$R^\phi{}_{\theta\theta\phi} = -1. \tag{13}$$

Hence, we have, for the acceleration of s in the direction of ϕ,

$$A^\phi = R^\phi{}_{\theta\theta\phi} T^\theta T^\theta S^\phi = 1 \tag{14}$$

and for the acceleration of s in the direction of θ,

$$A^\theta = R^\theta{}_{\theta\theta\phi} T^\theta T^\theta S^\phi = g^{\theta\theta} R_{\theta\theta\theta\phi} T^\theta T^\theta S^\phi = 0. \tag{15}$$

On a sphere at the equator, the equator is itself a geodesic. But, if there exists some vector at a non-equitorial parallel of latitude, the geodesic points in some direction that is NOT the parallel of latitude. Hence, the vector that points along the geodesic no longer points strictly in the ϕ direction—as is the case a the equator—and is thus parameterized differently.

Let's look at something like the acceleration. Let's say that instead of the metric g we have

$$g = \overset{o}{g} + \hbar. \tag{16}$$

The the acceleration can be written in terms of the acceleration in $\overset{o}{g}$ plus some acceleration in t.

$$T^\mu(\overset{g}{\nabla}_\nu)T^\nu = T^\mu(\overset{\overset{o}{g}}{\nabla}_\nu)T^\nu + A^\mu(\hbar) \tag{17}$$

2 A brief journey into Carroll's appendices

We start in Appendix A and move onto Appendix B in the next lecture. Appendix A concerns maps between manifolds. Suppose we have two manifolds M and N and a real space R. Then there are functions from each of these to the others according to Figure 2.

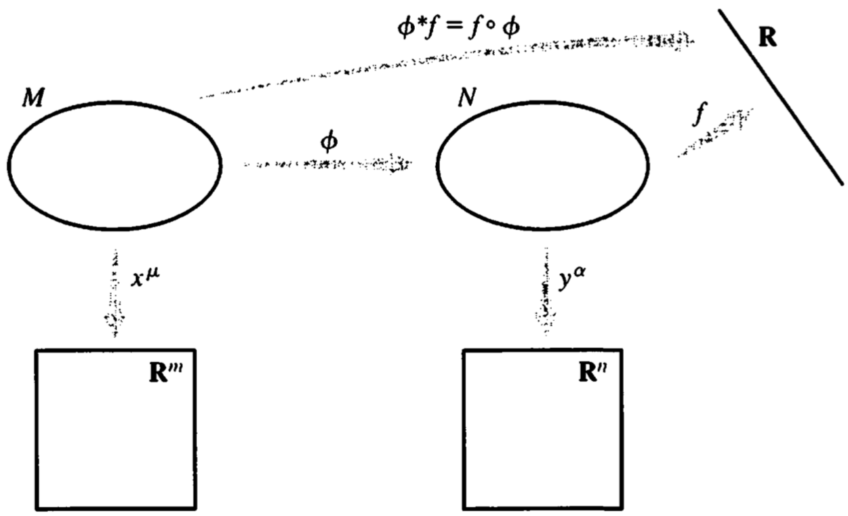

FIGURE A.1 The pullback of a function f from N to M by a map $\phi : M \to N$ is simply the composition of ϕ with f.

$$\phi : x^\mu \to y^\alpha(x^\mu) \tag{18}$$

Let's look at the cases where $m \neq n$, i.e. where M is the 2-sphere and N is the 3-sphere. The composition maps M to R. And if we can map functions, then we can map vectors.

$$V = V^\mu \partial_\mu f(y^\alpha(x^\mu)) \tag{19}$$

$$= V^\mu \frac{\partial y^\alpha}{\partial x^\mu} \partial_\alpha f \tag{20}$$

$$(\phi_* V)^\alpha = \frac{\partial y^\alpha}{\partial x^\mu} V^\mu \tag{21}$$

This is called a *push forward*, while ϕ^* is a *pull back*. We can push forward vector fields and pull back functions. With the functions, we can pull back one-forms.

$$(\phi^* \omega)_\mu = \frac{\partial y^\alpha}{\partial x^\mu} \omega_\alpha \tag{22}$$

We can also pull back a metric.

$$g_{\mu\nu} = (\phi_* g_{\alpha\beta})_{\mu\nu} \tag{23}$$

An example Let's map $\theta, \phi \to x, y, z$. Then the map in 3-space will be the unit sphere with $0 < \theta < \pi$ and $0 < \phi < 2\pi$ according to

$$\begin{aligned} x =& 1 \sin\theta\cos\phi & (24) \\ y =& 1 \sin\theta\sin\phi & (25) \\ z =& 1 \cos\theta. & (26) \end{aligned}$$

$$\begin{aligned} \frac{\partial y^\alpha}{\partial x^\mu} &= \begin{pmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial y}{\partial\theta} & \frac{\partial z}{\partial\theta} \\ \frac{\partial x}{\partial\phi} & \frac{\partial y}{\partial\phi} & \frac{\partial z}{\partial\phi} \end{pmatrix} & (27) \\ &= \begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} & (28) \end{aligned}$$

Then we would like to calculate

$$\begin{aligned} g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta} & \quad (29) \\ \left[\frac{\partial y^\alpha}{\partial x^\mu} g_{\alpha\beta} = X_{\alpha\beta}\right]\frac{\partial y^\beta}{\partial x^\nu} & \quad (30) \end{aligned}$$

To do the first part, find

$$\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

This gives the same thing back but with β for the columns and μ for the rows. Next, mutiply it by itself but with ν for the rows.

$$\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix}\begin{pmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \tag{32}$$

We usually do coordinate transformations in n-dimensions then restrict. Here we didn't even use 3D. The 2D surface in fact lives somewhere else entirely.

What we have done here is a *pullback*.