Special and General Relativity, Fall 2018

Yang Yang and Sanjib Katuwal

September 24, 2018

1 Einstein Tensor

Bianci identity:

$$\nabla_\mu R_{\nu\rho\sigma\tau} + \nabla_\nu R_{\rho\mu\sigma\tau} + \nabla_\rho R_{\mu\nu\sigma\tau} = 0 \tag{1}$$

if we multiply the bianci identity above by metric matrix,then we can find:

$$\begin{aligned} & g^{\mu\sigma} g^{\rho\tau} (\nabla_\mu R_{\nu\rho\sigma\tau} + \nabla_\nu R_{\rho\mu\sigma\tau} + \nabla_\rho R_{\mu\nu\sigma\tau}) = 0 \\ & \Rightarrow \nabla^\mu R_{\mu\nu} - \nabla_\nu R + \nabla^\mu R_{\mu\nu} = 0 \end{aligned} \tag{2}$$

Where R is the Ricii scalar.
Then we can define the Einstein Tensor $G_{\mu\nu}$:

$$G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu} R \tag{3}$$

Use E-H action, we can derive:

$$\begin{cases} \frac{\delta S_g}{\delta g_{\mu\nu}} \Rightarrow \nabla_\mu G^\mu_\nu = 0 \\ \frac{\delta S_m}{\delta g_{\mu\nu}} \Rightarrow \frac{8\pi G}{c^p} \nabla_\mu T^\mu_\nu = 0 \end{cases} \tag{4}$$

where T^μ_ν is the energy-momentum tensor.

2 Killing Vector

Let's consider the motion in space-time:
The action is $I = 1/2mg_{\mu\nu}\dot{x^\mu}\dot{x^\nu}$, and the momentums are $P_\mu = mg_{\mu\nu}\dot{x^\nu}$.

$$\frac{dP_\mu}{d\tau} = 1/2m\partial_\mu g_{\rho\sigma}\dot{x^\rho}\dot{x^\sigma} \tag{5}$$

This equation is another form of geodesic equation, if $\partial_\mu g_{\rho\sigma} = 0$, then P_μ is constant. In sphere coordinates, since the metric matrix doesn't depend on ϕ and t, we can conclude that $P_t, P_\phi = L_z$ are conserved.

If the metric matrix is independent of some coordinate μ^*, the corresponding momentum $P_{\mu*}$ is conserved. $\mathbb{P} = P_{\mu*}e^{\mu^*}$, here we don't sum the index over,in general $P_{\mu*}$ is in some particular direction K^μ, so $P_{\mu*} = K^\mu P_\mu = K_\mu P^\mu$.This $P_{\mu*}$ is a scalar. Then, we can define the killing vectors:

$$\mathbb{K} = K^\mu\partial_\mu \tag{6}$$

If the quantity $P_{\mu*}$ is a constant along the path of motion, we have:

$$\frac{dP_{\mu*}}{d\tau} = 0 \leftrightarrow P^\mu\nabla_\mu(P^\nu K_\nu) = 0 \tag{7}$$

Expanding the expression above, we have:

$$\begin{aligned} P^\mu\nabla_\mu(P^\nu K_\nu) &= K_\nu P^\mu\nabla_\mu P^\nu + P^\mu P^\nu\nabla_\mu K_\nu \\ &= P^\mu P^\nu\nabla_\mu K_\nu \\ &= P^\mu P^\nu\nabla_{(\mu}K_{\nu)} \end{aligned} \tag{8}$$

In the second line above, we use the geodesic equation $P^\mu\nabla_\mu P^\nu = 0$, and in the third line $\nabla_{(\mu}K_{\nu)}$ means the symmetric part of $\nabla_\mu K_\nu$, since $P^\mu P^\nu$ is a symmetric matrix, only the symmetric part of $\nabla_\mu K_\nu$ will contribute. If $\nabla_{(\mu}K_{\nu)} = 0$, then $K_\nu P^\nu$ is conserved along the trajectory.

Thus, we find the Killing's equation:

$$\nabla_{(\mu}K_{\nu)} = 0 \tag{9}$$

The solution to this equation K_ν is called killing's vector.

In 3-D flat space, we have 6 conserved momentum $P_x, P_y, P_z, L_x, L_y, L_Z$,thus we should have 6 killing's vectors:

$$\begin{cases} X^\mu = (1,0,0) \\ Y^\mu = (0,1,0) \\ Z^\mu = (0,0,1) \\ R^\mu = (-y,x,0) \\ S^\mu = (z,0,-x) \\ T^\mu = (0,-z,y) \end{cases} \tag{10}$$

In 4-D space, the maximum killing vectors are 10. In Minkowski space, $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$, it has 10 killing vectors, it has 4 translation killing vectors, 3 rotation killing vectors and 3 boost killing vectors B_x, B_y, B_z:

$$B_x = x\partial_t + t\partial_x, etc. \tag{11}$$

Flat Minkowski space is Maximally symmetric whose Ricci scalar is 0.
In n-D, there exits maximally symmetric spaces with $R > 0, const; R = 0; R < 0, const.$. The space with constant positive R is n-sphere, and is called the de-sitter space. The space with negative constant curvature is the hyperbolic space or the anti de-sitter space.

3 geodesic separation

Let's consider 2-D first.In flat 2-D space, suppose $\gamma_s(t)$ is a one-parameter family of geodesic,for each $s \in \mathbb{R}$, γ_s is a geodesic parameterized by the affine parameter t. Let γ_1, γ_2 to be 2 geodesic, then the geodesic separation stays the same.

Now, let's consider non-flat space. We are looking at the separation of two geodesic. As we go up with the parameter t, we have the same parameter t in both geodesic, then the geodesic separation is not a constant. We can derive a equation for geodesic separation.