Special & General Relativity I

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Exponential Map

Consider a set

$$\mathcal{E} = \{(p, x_p)|\gamma(t; p, x_p), \gamma \in [0, 1]\}$$

where by definition, $\mathcal{E}$ is the tangent vector to the manifold at point p. Geodesics are used to create **exponential maps**, which are used to map the tangent space T_p of a point p to a region of the manifold that contains p. The definition of an exponential map is

$$\exp : \mathcal{E} \to M, \quad (p, x_p) \mapsto \exp_p(x_p) = \gamma(1; p, x_p) \text{ for } \gamma \in (0, 1)$$

For any $p \in M$ and any vector x_p in $\mathcal{E}$, there exists a unique geodesic $\gamma(t) = \gamma(t; p, x_p)$ in M such that

$$\gamma(0) = p, \ \dot{\gamma}(0) = x_p.$$

Recall that a linear parameterization of a geodesic is again a geodesic.

$$(p, x_p) \in \mathcal{E} \to (p, \lambda x_p) \in \mathcal{E}$$

x_p is a place in the tangent space T_pM and for each point in the tangent space, there is a vector that points from the origin to that point. So if we are given x_p, there is a curve γ in M that depends smoothly on p and x_p.

Characterizing Curvature

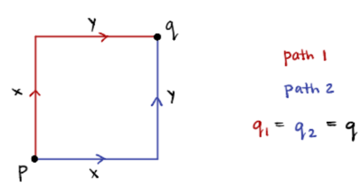

Figure 1: Translation from point p via two different similar paths leads to the same endpoint q in flat space.

Consider traveling from point p to point q in flat space via two different but similar paths as shown in Figure . It can easily be seen that the two paths will end at the same point q in flat space. This is also true that parallel transportation of a vector via these two different paths will return the same vector at the point q. Therefore, parallel transport of $V_p \to V_q$ is independent of path in flat space.

On a sphere, however, if the path is long enough then this same procedure will cause path 1 and path 2 to end at two different points q_1 and q_2 respectively where $q_1 \neq q_2$. This works if the path is long enough but what about the case of motion in the neighborhood of p? We can imagine an

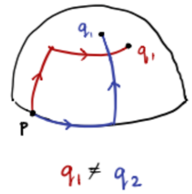

Figure 2: Translation from point p via two different similar paths leads to the same endpoint q in flat space.

alternative measure in which a grid is built out of parameter intervals s and t along which we will end up at the same point. We will not, however, end up with the same vector. This provides some measure of the curvature. We can measure the changes in the vector along these paths.

$$[\vec{X},\vec{Y}]\vec{V} \to \delta V^\sigma \sim (\nabla_\mu \nabla_\nu V^\sigma - \nabla_\nu \nabla_\mu V^\sigma) X^\mu Y^\nu$$

The commutator of two covariant derivatives measures the difference between parallel transporting the vector first one way and then the other, versus the opposite ordering. Thus

$$(\nabla_\mu \nabla_\nu V^\sigma - \nabla_\nu \nabla_\mu V^\sigma) = R^\sigma{}_{\rho\mu\nu} V^\rho$$

provides a proper measure of the curvature (disregarding any torsion).

The **Riemann tensor** is defined to be

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma{}_{\nu\rho} - \partial_\nu \Gamma^\sigma{}_{\mu\rho} + \Gamma^\sigma{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\lambda} - \Gamma^\sigma{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\rho}$$

and it is antisymmetric in the first two indices and the last two indices

$$R^\sigma{}_{\rho\mu\nu} = -R^\sigma{}_{\rho\nu\mu}$$

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$$

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$$

and it is symmetric upon exchange of the pairs

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}.$$

The sum of cyclic permutations of the last three indices vanishes

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$$

Example

Consider a sphere of radius a with metric

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2.$$

Two Christoffel symbols that we wrote last time

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{\cos\theta}{\sin\theta}$$

A promising component of the Riemann tensor is

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda}R^{\lambda}_{\phi\theta\phi} = g_{\theta\theta}R^{\theta}_{\phi\theta\phi} = a^2\sin^2\theta$$

We can compute the **Ricci tensor** via

$$R_{\mu\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$$

giving us

$$R_{\theta\theta} = g^{\phi\phi}R_{\theta\phi\theta\phi} = 1$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta}R_{\theta\phi\theta\phi} = \sin^2\theta$$

We can easily compute the **Ricci scalar**, which completely characterizes the curvature

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{a^2}$$

From this computation we can see two points:

1. $R > 0$ for a sphere.
2. $R \,\alpha\, 1/a^2$ so the Ricci scalar has units $[R] = L^{-2}$

Differentiate $\nabla_\tau R_{\mu\nu\rho\sigma}$ to obtain three derivatives of the metric. The derivatives share the same property where the sum of the cyclic permutations vanishes

$$\nabla_\tau R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\tau\mu\rho\sigma} + \nabla_\mu R_{\nu\tau\rho\sigma} = 0.$$

Due to the antisymmetric nature of the Riemann tensor, we can write this as

$$\nabla_{[\lambda}R_{\mu\nu]\rho\sigma} = 0$$

which is known as the **Bianchi identity** and is closely related to the Jacobi identity.
Next time we will discuss the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$$

and see that

$$\nabla^\mu G_{\mu\nu} = 0$$