PHZ 6607 Special and General Relativity
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1 Covariant Derivatives

The covariant derivative of a vector field is given by

$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\sigma\nu} V^\sigma \tag{1}$$

where

$$\Gamma^\mu_{\sigma\nu} = \frac{1}{2} g^{\mu\lambda}(\partial_\sigma g_{\nu\lambda} + \partial_\nu g_{\lambda\sigma} - \partial_\lambda g_{\sigma\nu}) \tag{2}$$

is the Christoffel symbol, which is a unique connection constructed from the metric that gives a way to relate vectors in the tangent space of near by points. It is worth while to note that the covariant derivative is a (1,1) tensor, while each component of the derivatives on their own are not tensors. This can be proven by showing that they do not transform as a tensor would, while their sum does.

The covariant derivative is necessary because we need an operation that reduces to a partial derivative in flat space but has the tensoral properties necessary to describe the curvature of a given manifold. The form of the covariant derivative is elucidated by remembering that V^ν is really only the components of a vector and we must also consider the variation of the basis as we move along tangent spaces in the manifold:

$$\vec{V} = V^\mu \vec{e}_\mu \tag{3}$$

$$d\vec{V} = (dV^\mu)\vec{e}_\mu + V^\mu (d\vec{e}_\mu) \tag{4}$$

So considering a derivative with respect to x^ν

$$\frac{\partial \vec{V}}{\partial x^\nu} = \frac{\partial V^\mu}{\partial x^\nu}\vec{e}_\mu + V^\mu \frac{\partial \vec{e}_\mu}{\partial x^\nu} \tag{5}$$

$$= \frac{\partial V^\mu}{\partial x^\nu}\vec{e}_\mu + V^\mu \boxed{\frac{\partial \vec{e}_\mu}{\partial x^\nu}} \to \Gamma^\sigma_{\mu\nu}\vec{e}_\sigma \tag{6}$$

$$= \left(\frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\sigma\nu} V^\sigma \right) \vec{e}_\mu \tag{7}$$

$$= \nabla_\nu V^\mu \vec{e}_\mu \tag{8}$$

In the case of scalars the covariant derivative reduces to the usual partial derivative.

2 Curved vs. Flat Spaces

Some surfaces may seem to have curvatures but are in fact a flat space. For two dimensions, the simple triangle test can show the distinction between a flat and curved surface. If the sum of the angles within a triangle is π, then the surface is flat. If the sum of the angles within a triangle is not π, then the surface has curvature.

Flat surface include of course a square, as well as shapes such as cylinders and torus because these surfaces can be mapped to a flat square. A simple example of a surface with curvature is a sphere.

3 Lorentz Transformations in Flat Space

Consider Corey who comes running in late to class with a meter stick (probably the meter stick's fault). In this one dimensional example Corey is running with his meter stick and it impacts Bernard at t=0 in his frame and Corey sees it impact at t' = 0. The relative measurements in the two frames are related by :

$$x' = \gamma(x - vt) \tag{9}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{10}$$

So in Corey's frame he is at $x' = -1m$ when the stick hits Bernard at t=0, implying:

$$x = \frac{x'}{\gamma}, \tag{11}$$

meaning that the spatial distance between Corey and Bernard at the meter sticks impact is different in their different frames.

Motion along the spatial axis as time moves forward defines a worldline. In general we consider tangents to worldlines pointing in the positive time direction. In conventional spacetime diagrams up is the future, down in the past, and anywhere else is elsewhere. Spacetime is made up events but ordering of the events in 'elsewhere' is not unique.