# Special and General Relativity Notes

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## 1 Perturbations of the Kerr Black Hole

Metric (Bayon-Lindquist coordinates):

$$ds^2 = \frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\sin^2\theta\Sigma^2}{\rho^2}\left(d\phi - \frac{2Mar}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \tag{1}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2mr + a^2$, $\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$, and $\Delta - a^2\sin^2\theta = 0$ gives the ergosphere.

Killing vectors:

$$\Phi = \partial_\phi \tag{2}$$
$$\mathbb{T} = \partial_t \text{ (not time like everywhere)} \tag{3}$$
$$\mathbb{K}^\mu = T^\mu + \Omega_H\phi^\mu \text{ (time-like on or near event horizon)} \tag{4}$$

Kerr's Approach:

$$g_{\mu\nu} = \eta_{\mu\nu} + fk_\mu k_\nu. \tag{5}$$

$k_\mu$ is null when:

$$f = (Gr^2 * 2Mr)/(r^2 + a^2z^2). \tag{6}$$

r is given by:

$$1 = \frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2}. \tag{7}$$

When you put M = 0, $g_{\mu\nu} = \eta_{\mu\nu}$ or flat space in spheroidal polar coordinates. This was originally used in radar development. The scale wave equations in this space are separable.

Spheroidal polar coordinates (are separable): $\Box\psi = 0$ and $\nabla_\mu F^{\nu\mu} = j^\nu$. $K_{\mu\nu}$ satisfies extended Killings equations ($K_{(\mu\nu;\zeta)}$).

$$l^\mu = \frac{1}{\Delta}\left(r^2+a^2, \Delta, 0, a\right) \tag{8}$$
$$n^\mu = \frac{1}{2\rho^2}(r^2+a^2, -\Delta, 0, a) \tag{9}$$

Leading to:

$$K^{\mu\nu} = 2\rho^2 l^{(\mu} n^{\nu)} + r^2 g^{\mu\nu} \tag{10}$$

When $K^{\mu\nu} p_\mu p_\nu = C$. Thus we have E, L, c, and m as conserved quantities. This means that the geodesic equations (and Hamilton-Jacobi equations) are separable. This separability implies the separability of the scalar wave equation ($\Box\psi = 0$), Maxwell's equations ($\nabla_\mu F^{\nu\mu} = j^\nu$), Dirac equation, ..., including the perturbation equation's for the spin $\pm 2$ fields.

$$g^{\mu\nu} = f_1 l^{(\mu} n^{\nu)} + f_2 m^{(\mu} m^{\nu)} \tag{11}$$

$$m^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}(ia\sin\theta, 0, 1, i\csc\theta) \tag{12}$$

if a = 0 the last two parts become $(\partial_\theta + i/\sin\theta\partial_\phi)$

$$\partial_r \Delta \partial_r - \frac{1}{\Delta}\left((r^2+a^2)\partial_t + a\partial_\phi - (r-M)s\right)^2 - 4sr\partial_t +$$
$$\frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} + \frac{1}{\sin^2\theta}\left(a^2\sin^2\theta\partial_t + \frac{\partial}{\partial\phi} + i\cos\theta s\right)^2 - 4ias\cos\theta\partial t \tag{13}$$

When this all acts on a scalar field $\Psi_s$ this equates to zero.

| | |
|---|---|
| s = 0 | Scalar |
| s = ±1/2 | Dirac |
| s = ±1 | Maxwell |
| s = ±3/2 | Rarita-Schwinger |
| s = ±2 | Gravity |

In this form the first half does not depend on theta, the second does not depend on r meaning that it is indeed separable.

1. Separability requires some background.
2. All these fields (for each s) are "gauge" invariant ("gauge" is equivalent to gauge and local tetrad here)

   Maxwell: $A_\mu$ is the potential.

   $F^{[\mu\nu]}$ has 6 degrees of freedom and this maps to $\varphi_{0,1,2}$ which is complex and again has 6 degrees of freedom. $\varphi_0$ corresponds to $s = +1$ and $\varphi_2$ corresponds to $s = -1$.

   Gravity: $g^{\mu\nu}$ is the potential.

   $R^\mu_{\nu\rho\sigma}$ is split into $G^{\mu\nu}$ (10 d.o.f.) and $C^\mu_{\nu\rho\sigma}$ (10 d.o.f.).

   $C^\mu_{\nu\rho\sigma}$ (Weyl Tensor) can be mapped to $\Psi_{0,1,2,3,4}$. $\Psi_0$ corresponds to $s = +2$ and $\Psi_4$ corresponds to $s = -2$. Can calculate these in Schwarzschild. $\Psi_0$ and $\Psi_4$ are 0 in Schwarzschild and Kerr Metric. $\Psi_2 \neq 0$ in Schwarzschild and Kerr Metric.

3. All Black Hole Binary Mergers result in Kerr Black Holes meaning they have angular momentum ($a/M \sim 0.7$).

$$g_{\mu\nu}^{\text{phys}} = g_{\mu\nu} + h_{\mu\nu} \tag{14}$$

where $h_{\mu\nu}$ is the strain and is roughly $\Delta l/l$ and is not gauge invariant.

In numerical relativity you can not write down $h_{\mu\nu}$ in their coordinates instead you compute $\Psi_0$, $\Psi_4$ (gauge invariant)

$$\Psi_{0,4} \sim (h_+ \pm i h_x)^{\cdot\cdot} \text{ (T-T gauge)} \tag{15}$$

$$h_+ \pm i h_x = \int \int \Psi_{0,4} dt' dt \tag{16}$$