Special and General Relativity-I, Fall 2018

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Geodesics of Schwarzschild

In the spherical coordinates, the Schwarzschild metric can be written as:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

where $d\Omega^2$ is the metric on a unit two-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{2}$$

We want to calculate the geodesics for this metric using the principle of variation or the Lagrangian approach. We can define the Lagrangian as:

$$\mathrm{L} = \frac{1}{2} m g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{3}$$

where λ is a parameter, not necessarily the proper time τ. Therefore, using Eq.1, we get,

$$\mathrm{L} = \frac{1}{2} m \left[-\left(1 - \frac{2GM}{rc^2}\right) c^2 \dot{t}^2 + \dot{r}^2 \left(1 - \frac{2GM}{rc^2}\right)^{-1} + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2 \right] \tag{4}$$

where the derivative "$\cdot$" indicates total derivative with respect to our parameter λ. For this Lagrangian, the conjugate momenta are:

$$p_t = \frac{\partial L}{\partial \dot{t}} = -mc^2 \left(1 - \frac{2GM}{rc^2}\right) \dot{t} = -E \text{ (a constant)} \tag{5}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta \dot{\phi} = L_z \text{ (a constant)} \tag{6}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \left(1 - \frac{2GM}{rc^2}\right)^{-1} \tag{7}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} \tag{8}$$

Please note that the constancy in Eqs.5 and 6 is implied by the fact that the coordinates (t, ϕ) are cyclic. One additional thing we need is the following relation:

$$\begin{aligned} & p^\mu p_\mu = -m^2 c^2 \\ \text{or,} \quad & p_\mu p_\nu g^{\mu\nu} = -m^2 c^2 \\ \text{or,} \quad & -\frac{p_t^2}{c^2} \left(1 - \frac{2GM}{rc^2}\right)^{-1} + \left(1 - \frac{2GM}{rc^2}\right) p_r^2 + \frac{1}{r^2} \left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) = -m^2 c^2 \end{aligned} \tag{9}$$

Because the problem is spherical symmetric, to simplify our calculation we choose frame such that the orbit lies on the equitorial plane so that $\dot{\theta} = 0$ and, therefore, L_z gives the total angular moementum, L, which is the term in the last bracket of LHS in the above equation. Therefore, using Eqs.(5,6) we have,

$$-\frac{E^2}{c^2} \left(1 - \frac{2GM}{rc^2}\right)^{-1} + \left(1 - \frac{2GM}{rc^2}\right) p_r^2 + \frac{L^2}{r^2} = -m^2 c^2 \tag{10}$$

We can solve this equation for p_r and using Eqn.(7), we can solve for $r(\lambda)$, which is equivalent to solving equation of geodesic. However, solving the equation

$$-c^2 \dot{t}^2 \left(1 - \frac{2GM}{rc^2}\right) + r^2 \dot{\phi}^2 = 0$$

and finding $\phi(t)$ is not equivalent to solving a geodesic equation.
Now, using Eq.(7) in Eq. (10) gives us the following equation,

$$\begin{aligned}\left(\frac{dr}{d\lambda}\right)^2 &= \frac{E^2}{m^2c^2} - \frac{1}{m^2}\left(1-\frac{2GM}{rc^2}\right)\left(m^2c^2+\frac{L^2}{r^2}\right) \\ &= \left(\frac{E^2}{m^2c^2}-c^2\right) + \left(\frac{2GM}{r}-\frac{L^2}{m^2r^2}\right) + \left(\frac{2GML^2}{m^2c^2r^3}\right)\end{aligned} \tag{11}$$

If we compare Eq.(11) with analogous Newtonian Equation, we see that the term in the first bracket can be replaced by Newtonian kinetic energy in the Newtonian limit, the second term is a Newtonian term and the final last term is actually a General relativistic effect.
Now, in what follows, we will take:

$$c = 1; \quad \frac{E}{m} = \tilde{E}; \quad \frac{L}{m} = \tilde{L}$$

so that Eq.(11) can be written as

$$\begin{aligned}\dot{r}^2 &= \tilde{E}^2 - \tilde{V}^2 \\ \text{where,} \quad \tilde{V}^2 &= \left(1-\frac{2GM}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)\end{aligned} \tag{12}$$

which has been taken in analogous to classical Newtonian theory.

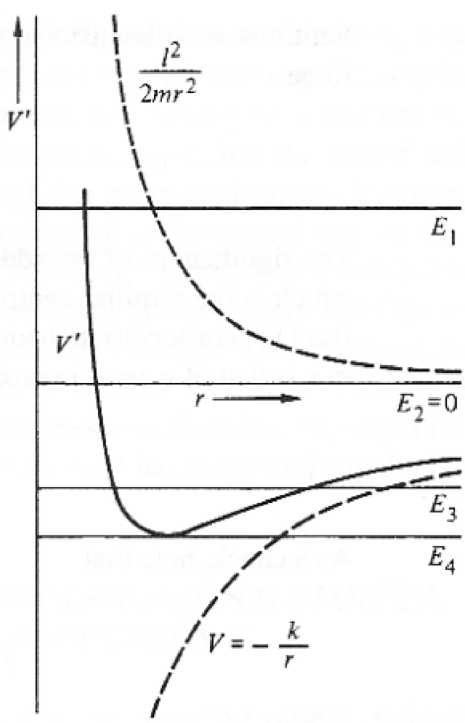

Figure 1: An equivalent one-dimensional effective potential for inverse square force law in classical Newtonian mechanics. Ref. *Classical Mechanics* by Goldstein *et. al.*

In such classical consideration, we can either have an orbit or scattering. For example, in the above figure, E_1 represents a scattering situation and E_3 represents an orbiting situation. If the particle is unbounded, the path are parabolic or hyperbolic and, the bounded orbits are either circles or ellipses. Also, there will be no circular orbits for massless particles. A similar consideration in General relativity for a massive particle would look like:

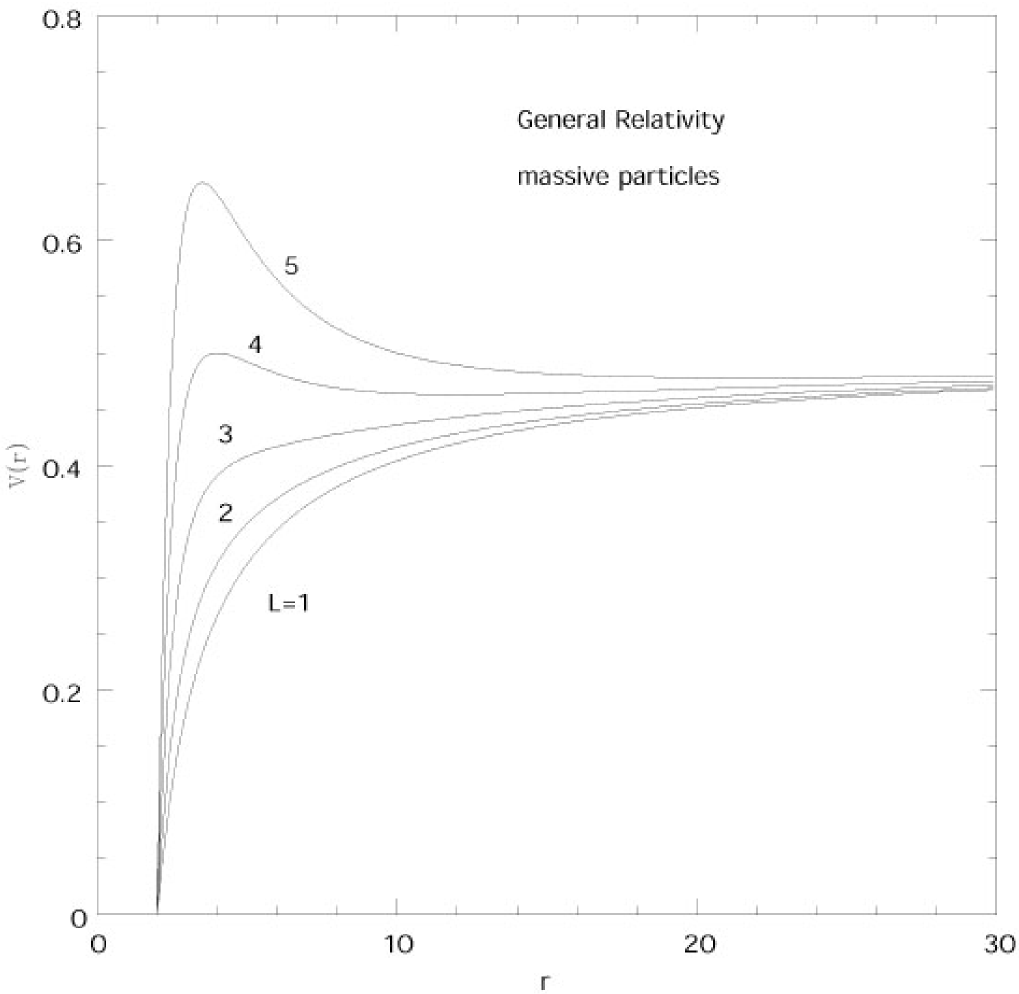

Figure 2: Effective Potential in GR, where L is angular momentum per unit mass and in units where $GM = 1$. Picture Credit: *Lecture notes in GR* by Sean Carroll.

At sufficiently large r, the two theories have similar predictions, but for sufficently small r, the situation in GR is different. For example as $r \to 0$, the potential goes to $-\infty$ instead of ∞ as in Newtonian case. Therefore, a particle with sufficiently large energy (say E_1, consider a straight line in Figure(2) with this energy E_1 above all the curves there), may be absorbed by the blackhole, which is in contrast to the result obtained in classical analysis. Another such non-classical feature is that at $r = 2GM$ (with c=1), the potential is always zero and inside this radius, there is a blackhole. I think rest of the discussion will be easier if we consider a particular value of L, which we do in Figure(3).

If a particle comes from infinity and has an energy corresponding to point G in the Fig(3), it is a case of scattering. Possibility of circular orbits occur at the maxima or minima , which is point A and B in Fig(3). Obviously, point A corresponds to an unstable orbit. Firstly, we will consider circular orbits for a massive particle a bit more quantitatively here. There will be two requirements for circular orbits:

$$\frac{dr}{d\lambda} = 0 \implies \tilde{V}^2 = \tilde{E}^2 \tag{13}$$

$$\frac{d^2r}{d\lambda^2} = 0 \tag{14}$$

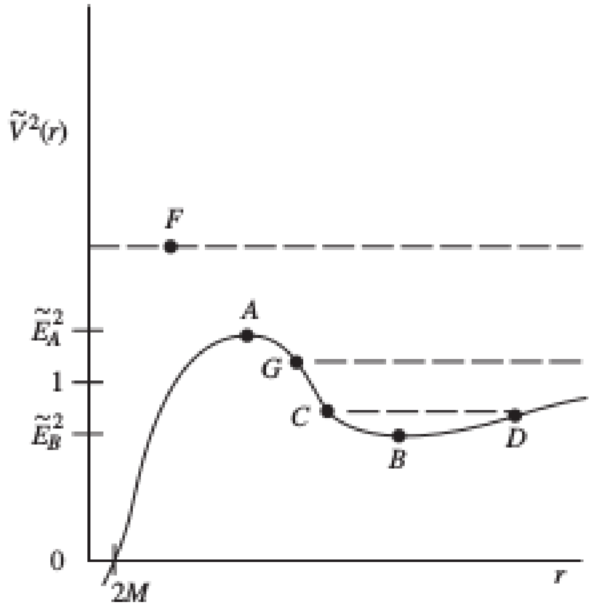

Figure 3: Typical effective potential for a massive particle of fixed specific angular momentum in the Schwarzschild metric. $G = 1$ Ref. *A first course in GR* by Schutz

Please note that the first condition is true for all turning points and is, therefore, not unique. Differentiating Eq.(12) with respect to λ and using the second of the above mentioned conditions would give us,

$$2GMr^2 - 2\tilde{L}^2 r + 6GM\tilde{L}^2 = 0 \tag{15}$$

which is quadratic in r and therefore,

$$r = \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12G^2M^2\tilde{L}^2}}{2GM} \tag{16}$$

It is evident from the above equation that for $\tilde{L}^2 > 12G^2M^2$, there are two values of r, for $\tilde{L}^2 = 12G^2M^2$, one value of r and if $\tilde{L}^2 < 12G^2M^2$, no such r exists. In the limit $\tilde{L} \to \infty$ Eq.(16) becomes,

$$r = \frac{\tilde{L}^2 \pm \tilde{L}^2\left(1 - 6GM/\tilde{L}^2\right)}{2GM} = \left(3GM, \frac{\tilde{L}^2}{GM}\right) \tag{17}$$

In this limit, the outer stable orbit goes farther and farther, and the inner unstable orbit goes to $3GM$. Also, the minimum value of r for stable circular orbits, with $\tilde{L}^2 = 12G^2M^2$, is $r = 6GM$. Therefore, it is evident that, for $3GM < r < 6GM$, the circular orbits are unstable and for $r > 6GM$, Schwarzschild metric has stable circular orbits. Beside circular bound orbits, there can also be bounded non-circular orbits, which oscillate around stable circular radius. For a massless particle, the equation analogous to Eq.(12) would be:

$$\begin{gathered} \dot{r}^2 = E^2 - V^2 \\ \text{with,}\;\; V^2 = \left(1 - \frac{2GM}{r}\right)\frac{L^2}{r^2} \end{gathered} \tag{18}$$

The figures analogous to Fig. (2,3) for massless particles are Figures (4,5).

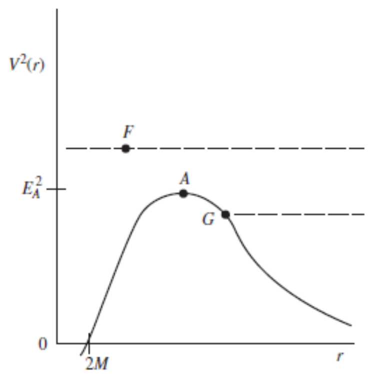

Figure 4: Typical effective potential for a massless particle of fixed specific angular momentum in the Schwarzschild metric. $G = 1$ Ref. *A first course in GR* by Schutz

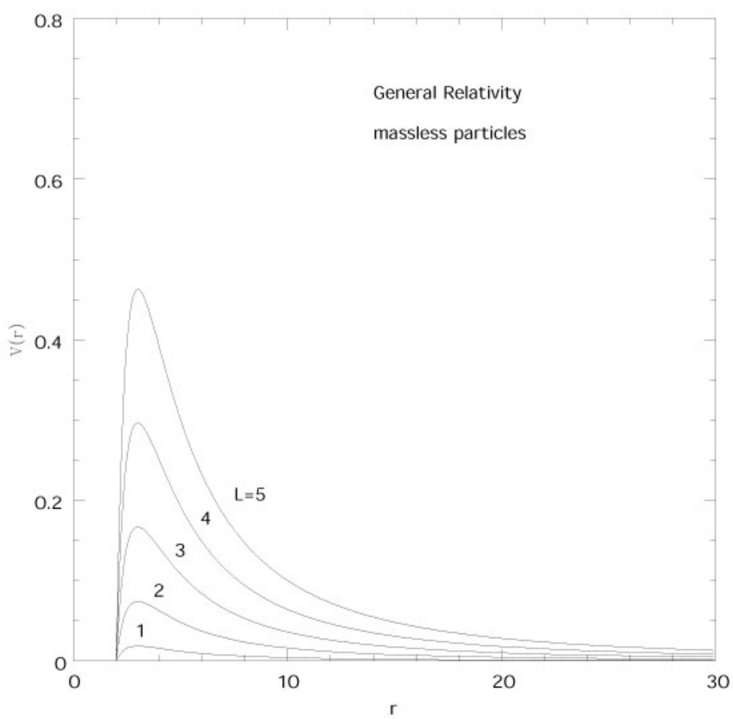

Figure 5: Effective Potential in GR, where L is angular momentum per unit mass and in units where $GM = 1$. Picture Credit: *Lecture notes in GR* by Sean Carroll.

Analogous calculation for massless particles will show that a massless particle will not have corresponding point B (see Figure(4)) so there can be no stable circular orbits for them. For them, the unstable circular orbit will always be at $r = 3GM$, which is the point of maxima in the potential (see Figure(5)). So, massless particles can move in a circular orbit forever with this radius, independent of L, but even a slight perturbation will make it go either to $r = \infty$ or $r = 0$.