Special and General Relativity I

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Action Principle

Using the metric as the dynamical variable, we can make scalars to use as the Lagrangian. The simplest independent scalar constructed from the metric is the Ricci scalar. Hilbert proposed this as the simplest choice for a Lagrangian (known as the Hilbert action):

$$S_H = \int d^4x\sqrt{-g}R(g^{\mu\nu}), \tag{1}$$

and we consider the behavior of S_H under small variations of the inverse metric using $R = g^{\mu\nu}R_{\mu\nu}$:

$$\delta S_H = \int d^4x\sqrt{-g}\left(g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} + R\frac{\delta\sqrt{-g}}{\sqrt{-g}}\right), \tag{2}$$

where we can call these three terms $(\delta S_H)_1$, $(\delta S_H)_2$, and $(\delta S_H)_3$, respectively.

Last week we saw that

$$\delta g = gg^{\mu\nu}\delta g_{\mu\nu} = -gg_{\mu\nu}\delta g^{\mu\nu}, \tag{3}$$

and if we look at $\delta g_{\mu\nu}$, we have

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}. \tag{4}$$

This comes from considering $\delta(g^{\mu\sigma}g_{\sigma\nu}) = 0$, since $g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$ (the Kronecker delta) is unchanged under any variation.

We also have that

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2}\frac{\delta(-g)}{\sqrt{-g}} \\ &= -\frac{1}{2}\frac{-g}{\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu},\end{aligned} \tag{5}$$

which contributes under the integral.

Considering the Riemann tensor:

$$R^\rho_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu), \tag{6}$$

we can do variations on the connection by replacing $\Gamma^\rho_{\nu\mu} \to \Gamma^\rho_{\nu\mu} + \delta\Gamma^\rho_{\nu\mu}$ and taking the covariant derivative:

$$\nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda(\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\lambda\nu}\delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu}\delta\Gamma^\rho_{\nu\sigma}. \tag{7}$$

It can be shown that

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\rho_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\rho_{\lambda\mu}), \tag{8}$$

and now we can express $(\delta S_H)_1$ as:

$$\begin{aligned}(\delta S_H)_1 &= \int d^4x\sqrt{-g}g^{\mu\nu}\left[\nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu})\right] \\ &= \int d^4x\sqrt{-g}\nabla_\sigma\left[g^{\mu\nu}(\delta\Gamma^\sigma_{\mu\nu}) - g^{\mu\sigma}(\delta\Gamma^\lambda_{\lambda\mu})\right]\end{aligned} \tag{9}$$

.

Plugging in the expression for $\delta\Gamma^\sigma_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$, which turns out to be

$$\delta\Gamma^\sigma_{\mu\nu} = -\frac{1}{2}\left[g_{\lambda\mu}\nabla_\nu(\delta g^{\lambda\sigma}) + g_{\lambda\nu}\nabla_\mu(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta}\nabla^\sigma(\delta g^{\alpha\beta})\right],$$

we then have

$$(\delta S_H)_1 = \int d^4x\sqrt{-g}\nabla_\sigma\left[g_{\mu\nu}\nabla^\sigma(\delta g^{\mu\nu}) - \nabla_\lambda(\delta g^{\sigma\lambda})\right].$$

The above is an integral with respect to the natural volume element of the covariant divergence of a vector. According to Stokes's Theorem, this is equal to a boundary term at infinity, which we can set to zero by making the variation vanish at infinity. Therefore, this term contributes nothing to the total variation. Thus, we are left with

$$\begin{aligned}\delta S_H &= \int d^4x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} \\ &= \int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu}.\end{aligned} \tag{10}$$

Energy Conditions

In order to understand properties of Einstein's equations that hold for a variety of different sources (not only for specific cases such as scalar fields, EM fields, etc), we need to impose energy conditions that limit the arbitrariness of $T_{\mu\nu}$. For instance:

$$\begin{aligned}T^{\mu\nu}(y^\sigma) &= m\int\left[\frac{\delta^4(y^\sigma - x^\sigma(\tau))}{\sqrt{-g}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau\right] \\ &= \left[\frac{m\delta^3(y^i - x^i(t))}{\sqrt{-g}u^t}\right]u^\mu u^\nu \\ &= \rho u^\mu u^\nu,\end{aligned} \tag{11}$$

where ρ is the energy density.

Let's consider matter that is composed of point particles in order to deal with the fact that when we begin to accelerate a large object by exerting a force on a particular side, then that side would tend to accelerate before the opposite side would, which would make our analysis much more complex. In this case, the stress tensor for perfect fluid is:

$$T_{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu}, \tag{12}$$

where u^μ is the fluid 4-velocity. In the case of electromagnetism, for example, $T^\mu_{\mu_{EM}} = 0$, which gives $p = 1/3\rho c^2$.

The problem is that it is not clear how generic this stress tensor structure is, e.g. how it appears in the rest frame, and whether the assumption of a perfect fluid is a solid assumption for any general case. Are there ways to get stress tensors that do not depend on being a perfect fluid? Also, there are an infinite number of metrics that obey Einstein's equation: simply compute $G_{\mu\nu}$ for a certain metric and then set $T_{\mu\nu} = G_{\mu\nu}$, however, we are looking for solutions for "realistic" sources of energy. A way to do that is by imposing energy conditions, which are coordinate-invariant restrictions on the energy-momentum tensor. There are various possible energy conditions; we list some of them below.

Weak Energy Condition (WEC)

The WEC states that

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq 0, \tag{13}$$

where t^{μ} is any time-like vector.

This implies that $\rho \geq 0$ and $\rho + p \geq 0$. This allows negative pressure, as can be seen in the Figure on the next page.

Null Energy Condition (NEC)

The NEC states that

$$T_{\mu\nu}l^{\mu}l^{\nu} \geq 0, \tag{14}$$

where l^{μ} is any null vector.

This also implies $\rho + p \geq 0$ as the NEC is a special case for the WEC. The energy density may be negative as there may be a compensating positive pressure.

Dominant Energy Condition (DEC)

DEC is the WEC with an additional requirement: $T^{\mu\nu}t_{\mu}$ must be a nonspacelike vector, i.e.,

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq 0$$

and

$$T_{\mu\nu}T^{\nu}_{\lambda}t^{\mu}t^{\lambda} \leq 0.$$

For a perfect fluid, this means: $\rho \geq |p|$. So in this case, the energy density cannot be negative and greater than (or equal to) the pressure.

Null Dominant Energy Condition (NDEC)

Special case of the DEC, the NDEC only applies to null vectors. So we have

$$T_{\mu\nu}l^{\mu}l^{\nu} \geq 0,$$

where $T^{\mu\nu}l_{\nu}$ is a nonspacelike vector. In this case, negative densities are allowed so long as $p = -\rho$, meaning that it does not exclude vacuum energy.

Strong Energy Condition (SEC)

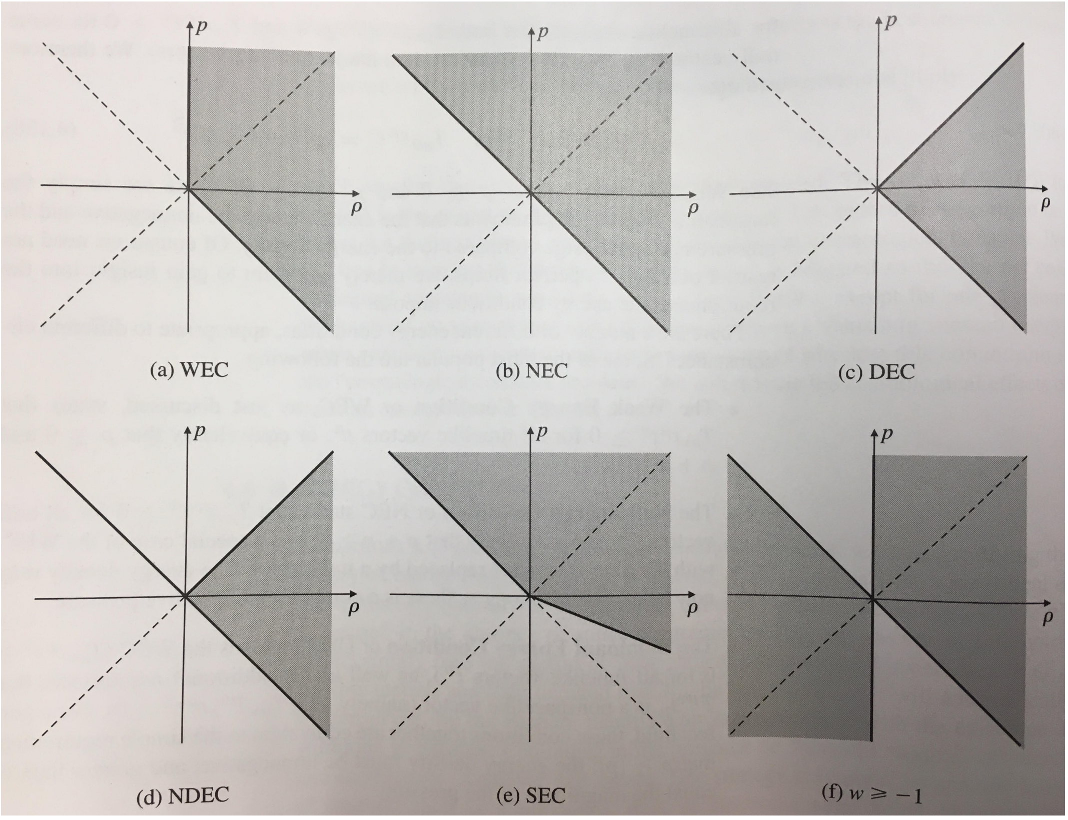
The SEC states that

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq \frac{1}{2}T^{\lambda}_{\lambda}t^{\sigma}t_{\sigma},$$

for any timelike vector t^{μ}. It means that $\rho + p \geq 0$ and $\rho + 3p \geq 0$.

The SEC implies the NEC, excluding excessively large negative pressures. Also, it implies that gravitation is attractive.

The figure in the next page illustrates all these energy conditions, including also a plot for the $w \geq -1$ case ($w = p/\rho$ is the equation-of-state parameter, a useful concept in cosmology).



Source: Carroll 2004, Spacetime and Geometry, Figure 4.3.