General Relativity

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The Killing vector on a sphere is equal to (0,1)

$$L^z = (0,1) = p^\phi$$

This is the Killing vector and its index is up.
To satisfy Killing's equation, $\nabla_{(\mu}K_{\nu)} = 0$, we need to lower the index.

$$K_\mu = (0, \sin^2\theta)$$

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu$$

Three sets of components; $\theta\theta$, $\theta\phi$, $\phi\phi$.
So we have

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \cot\theta$$

$\theta\theta$ component:

$$\partial_\theta K_\theta + \Gamma^\theta_{\theta\phi} K_\theta + \Gamma^\phi_{\phi\theta} K_\phi = 0 \qquad \text{Each term here is zero}$$

$\phi\phi$ component:

$$\partial_\phi K_\phi - \Gamma^\theta_{\phi\phi} K_\theta - \Gamma^\phi_{\phi\phi} K_\phi = 0 \qquad \text{Again, each term here is zero}$$

$\theta\phi$ component:

$$\partial_\theta K_\phi - \Gamma^\theta_{\theta\phi} K_\theta - \Gamma^\phi_{\theta\phi} K_\phi + \partial_\phi K_\theta - \Gamma^\theta_{\phi\theta} K_\theta - \Gamma^\phi_{\phi\theta} K_\phi = 2\sin\theta\cos\theta - \sin\theta\cos\theta - \sin\theta\cos\theta = 0$$

$$L^x = (-\sin\phi, -\cot\theta\cos\theta)$$
$$K_\mu = (-\sin\phi, -\sin\theta\cos\theta)$$

To prove this using Killing's theorem is not trivial. We proved it in a pretty trivial way here.

In 2D ($ds^2 = dr^2 + r^2 d\theta^2$),

$$\Gamma^r_{\theta\theta} = -r\,;\,\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$
$$R^r{}_{\theta r\theta} = \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^r_{r\theta} + \Gamma^r_{r\mu}\Gamma^\mu_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta}$$
$$= \partial_r \Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta} = -1 - (-r)\left(\frac{1}{r}\right) = -1 + 1 = 0$$

In flat space all 20 components of $R^\mu{}_{\nu\rho\sigma}$ are zero in *any* coordinate system.
Corollary: All 20 components are gauge invariant in flat space.

Derivatives

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\nu\mu} V^\nu$$

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right)$$
$$= \frac{1}{2} g^{\mu\sigma}\frac{\partial g_{\sigma\mu}}{\partial x^\nu} + \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\nu\mu}}{\partial x^\sigma} - \frac{\partial g_{\nu\mu}}{\partial x^\sigma}\right)$$
$$= \frac{1}{2} g^{\mu\sigma}\frac{\partial g_{\sigma\mu}}{\partial x^\nu}$$

There are three points to consider here.

1. $g = \tilde{\epsilon}^{\mu\nu\rho\sigma} g_{1\mu} g_{2\nu} g_{3\rho} g_{4\sigma} = \frac{1}{n!}\tilde{\epsilon}^{\mu\nu\rho\sigma}\tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\alpha\mu} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma}$ where n=4
2. $g^{\mu\alpha} = \frac{1}{g}\left(\frac{1}{(n-1)!}\tilde{\epsilon}^{\mu\nu\rho\sigma}\tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma}\right)$
3. $\partial g = \left(\frac{1}{(n-1)!}\tilde{\epsilon}^{\mu\nu\rho\sigma}\tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma}\right)\partial g_{\alpha\mu}$

∗ Example:

$$\frac{\partial g}{\partial x^\eta} = \left(\frac{1}{(n-1)!}\tilde{\epsilon}^{\mu\nu\rho\sigma}\tilde{\epsilon}^{\alpha\beta\gamma\delta} g_{\beta\nu} g_{\gamma\rho} g_{\delta\sigma}\right)\frac{\partial g_{\alpha\mu}}{\partial x^\eta} = g\, g^{\mu\alpha}\frac{\partial g_{\alpha\mu}}{\partial x^\eta}$$

• Now we can write,

$$\nabla_\mu V^\mu = \frac{1}{2} g^{\mu\sigma}\frac{\partial g_{\sigma\mu}}{\partial x^\nu} = \frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x^\sigma} = \partial_\mu V^\mu + \Gamma^\mu_{\nu\mu} V^\nu = \partial_\mu V^\mu + \frac{1}{\sqrt{-g}}\partial\nu(\sqrt{-g})V^\nu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,V^\mu)$$

Now consider

$$\int \sqrt{-g}\; d^4x\; \partial_\mu V^\mu$$

We know how to use Stoke's theorem in cartesian coordinates.
Covariantize to

$$\int \sqrt{-g}\; d^4x\; \nabla_\mu V^\mu = \int d^4x\; \partial_\mu(\sqrt{-g}\; V^\mu)$$

$$\frac{1}{2}\int \sqrt{-g}\; d^4x\; g^{\mu\nu}\; \partial_\mu\; \phi\; \partial_\nu\; \phi$$

$$-\int d^4x\left(\partial_\mu \sqrt{-g}\; g^{\mu\nu} \partial_\nu\; \phi\right)\delta\phi$$

With insertion of $\sqrt{-g}$ it can be written

$$-\int \sqrt{-g}\; d^4x\left(\frac{1}{\sqrt{-g}}\; \partial_\mu \sqrt{-g}\; g^{\mu\nu} \partial_\nu\; \phi\right)\delta\phi$$

The inside of the parentheses is

$$\nabla_\mu(g^{\mu\nu}\nabla_\nu\; \phi) = \nabla_\mu(V^\mu) = \frac{1}{\sqrt{-g}}\; \partial_\mu \sqrt{-g}\; g^{\mu\nu}\partial_\nu\; \phi = 0$$