Conditions under which the stress tensor for a point particle is conserved

We take the energy-momentum of the point particle to be given by:

$$T^{\mu\nu}(x) = m \int \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \dot{z}^\mu \dot{z}^\nu d\tau.$$

Then

$$\begin{aligned}
\nabla_\mu T^{\mu\nu}(x) =& \frac{\partial T^{\mu\nu}(x)}{\partial x^\mu} + \Gamma^\mu_{\mu\sigma}(x) T^{\sigma\nu}(x) + \Gamma^\nu_{\mu\sigma}(x) T^{\mu\sigma}(x) \\
=& m \int d\tau \Bigg[\frac{\partial \delta^4(x^\alpha - z^\alpha(\tau))}{\partial x^\mu} \frac{\dot{z}^\mu \dot{z}^\nu}{\sqrt{-g(z)}} \\
& + \Big(\Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \Big) \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \dot{z}^\sigma \Bigg] \\
=& m \int d\tau \Bigg[- \frac{d\delta^4(x^\alpha - z^\alpha(\tau))}{d\tau} \frac{\dot{z}^\nu}{\sqrt{-g(z)}} \\
& + \Big(\Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \Big) \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}} \dot{z}^\sigma \Bigg] \\
=& m \int d\tau \delta^4(x^\alpha - z^\alpha(\tau)) \Bigg[\frac{d}{d\tau} \Big(\frac{\dot{z}^\nu}{\sqrt{-g(z)}} \Big) \\
& + \Big(\Gamma^\mu_{\mu\sigma}(x) \dot{z}^\nu + \Gamma^\nu_{\mu\sigma}(x) \dot{z}^\mu \Big) \frac{\dot{z}^\sigma}{\sqrt{-g(z)}} \Bigg] \\
=& m \int d\tau \dot{z}^\sigma \Bigg[\nabla_\sigma \dot{z}^\nu + \Big(\Gamma^\mu_{\mu\sigma}(x) - \Gamma^\mu_{\mu\sigma}(z) \Big) \dot{z}^\nu \\
& + \Big(\Gamma^\nu_{\mu\sigma}(x) - \Gamma^\nu_{\mu\sigma}(z) \Big) \dot{z}^\mu \Bigg] \frac{\delta^4(x^\alpha - z^\alpha(\tau))}{\sqrt{-g(z)}}
\end{aligned}$$

The last two terms in square brackets are zero because the δ-function multiplies terms which vanish at coincidence — for **any** orbit $z^\alpha(\tau)$. Thus, if $z(\tau)$ is a geodesic, $\nabla_\mu T^{\mu\nu}(x) = 0$.

Note:

- The second equality holds because the derivative acts **only** on the δ-function.
- The third equality holds by introducing a change of sign upon swapping the arguments of the derivative on the δ-function and using $\dot{z}^\mu \partial/\partial x^\mu = d/d\tau$.
- The fourth equality holds from performing integration by parts.
- The fifth equality holds by rearrangement of terms after completing the covariant derivative on $\dot{z}^\nu$ and using $d \ln \sqrt{-g(z)}/d\tau = \Gamma^\mu_{\mu\sigma}(z) \dot{z}^\sigma$.
- The result given holds, and somewhat simpler, even if, from the beginning, $g(z) \Rightarrow g(x)$.