PHZ 6607, Special and General Relativity I

Class Number 21787, Fall 2018, Homework 4

Due at the start of class on Friday, November 30.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the equation:

$$R_{\alpha\beta\mu\nu} \equiv g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} = \frac{1}{2} \left(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu} \right),$$

for the Riemann tensor in locally Minkowski coordinates.

a) Verify the following identities:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}.$$

b) Verify the following identity:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0.$$

c) Show that the first set of equalities reduces the number of independent components of $R_{\alpha\beta\mu\nu}$ from $4\times4\times4\times4 = 256$ to $6\times7/2 = 21$. Hint: treat *pairs* of indices. Calculate how many independent choices of pairs there are for the first and the second pairs on $R_{\alpha\beta\mu\nu}$.)

d) Show that the second equation imposes only one further relation independent of those already established on the components, reducing the total of independent ones to 20.

2. Our Sun has an equatorial rotation velocity of about 2 km s^{-1} at its equator.

a) Estimate its angular momentum, on the assumption that the rotation is rigid (uniform angular velocity) and the Sun is of uniform density. As the true angular velocity is likely to increase inwards, this is a lower limit on the Sun's angular momentum.

b) If the Sun were to collapse to neutron-star size (say 10 km radius), conserving both mass and total angular momentum, what would its angular velocity of rigid rotation be? In nonrelativistic language, would the corresponding centrifugal force exceed the Newtonian gravitational force on the equator?

c) A neutron star of 1 $M_{\odot}$ and radius 10 km rotates 30 times per second (typical of young pulsars). Again in Newtonian language, what is the ratio of centrifugal to gravitational force on the equator? In this sense the star is slowly rotating.

d) Suppose a main-sequence star of 1 $M_{\odot}$ has a dipole magnetic field with typical strength 1 Gauss in the equatorial plane. Assuming flux conservation in this plane, what field strength should we expect if the star collapses to radius of 10 km? (The Crab pulsar's field is of the order of 10^{11} Gauss.)

3. This question concerns the massless scalar field and vacuum Electro-Magnetism.

a) Write out the Lagrangian for each.

b) Find the energy-momentum tensor for each.

c) Show that they each satisfy the dominant energy condition.

d) Show that they each satisfy the weak, null, and null dominant energy conditions.

e) Show that they each satisfy $w \geq -1$, where $w = p/\rho$ (*cf.* perfect fluid stress energy tensor).

4. Consider the orbit of massless particles, with affine parameter λ, in the equatorial plane of a Kerr black hole.

a) Show that:

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\Sigma^2}{\rho^4}\left(E - LW_{+}(r)\right)\left(E - LW_{-}(r)\right),$$

where $\Sigma^2 = (r^2 + a^2 0^2 - a^2\Delta(r)\sin^2\theta$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta(r) = r^2 - 2Mr + a^2$, E and L are the conserved energy and angular momentum, and you have to find the expressions for $W_{\pm}(r)$.

b) Using this result, and assuming $\Sigma^2 > 0$ everywhere, show that the orbit of a photon in the equatorial plane cannot have a turning point inside the outer event horizon r_{+}. This means that ingoing light rays cannot escape once they cross r_{+}, so it really is an event horizon.

5. Gravitational waves can be detected by monitoring the distance between two freely falling masses. Consider a simple Michelson interferometer with perpendicular arms, where the interferometer is composed of two perfectly reflecting end mirrors and a central 50/50 beam splitter being illuminated by monochromatic laser light.

a) How would you want to orient this interferometer to register the largest response from a plane gravitational wave of the form :

$$ds^2 = -dt^2 + [1 + A\cos(\omega(t - z))]dx^2 + [1 - A\cos(\omega(t - z))]dy^2 + dz^2?$$

b) By determining how long the laser light takes to traverse each arm, calculate the phase difference between the light from the two arms that will be observed at the output of the beam splitter (assume $A << 1$).

c) If each interferometer arm has mean length L, what is the largest change in phase difference at the beam splitter caused by the wave?

d) If each interferometer arm has mean length L, what is the largest change in arrival time caused by the wave?

e) What frequency ω (if any) would go undetected?