PHZ 6607, Special and General Relativity I

Class Number 21787, Fall 2018, Homework 3

Due at the start of class on Friday, November 9.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. Imagine that we have a *diagonal* metric, $g_{\mu\nu}$. Show that the Christoffel symbols are given by (*Note*: In these equations, $\mu \neq \nu \neq \sigma$, and repeated indices are **NOT** summed):

a) $$\Gamma^{\sigma}_{\mu\nu} = 0,$$

We have: $\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}\sum\limits_{\tau} g^{\sigma\tau}\left(\frac{\partial g_{\tau\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\tau}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\tau}}\right)$.

The first term vanishes unless $\tau = \sigma$ and $\tau = \nu \;\Rightarrow\; \sigma = \nu$.
The second term vanishes unless $\tau = \sigma$ and $\tau = \mu \;\Rightarrow\; \sigma = \mu$.
The third term vanishes unless $\tau = \sigma$ and $\mu = \nu$.
So, all terms vanish if $\mu \neq \nu \neq \sigma$.

b) $$\Gamma^{\sigma}_{\mu\mu} = -\tfrac{1}{2}(g_{\sigma\sigma})^{-1}\partial_{\sigma} g_{\mu\mu},$$

We have: $\Gamma^{\sigma}_{\mu\mu} == -\frac{1}{2}g^{\sigma\sigma}\frac{\partial g_{\mu\mu}}{\partial x^{\sigma}} = -\frac{1}{2}(g_{\sigma\sigma})^{-1}\partial_{\sigma} g_{\mu\mu}$ (no sum) as required.

c) $$\Gamma^{\sigma}_{\mu\sigma} = \partial_{\mu}\left(\ln\sqrt{|g_{\sigma\sigma}|}\right),$$

We have: $\Gamma^{\sigma}_{\mu\sigma} = \frac{1}{2}g^{\sigma\sigma}\frac{\partial g_{\sigma\sigma}}{\partial x^{\mu}} = \frac{1}{2}\left(g_{\sigma\sigma}\right)^{-1}\partial_{\mu} g_{\sigma\sigma} = \partial_{\sigma}\left(\ln\sqrt{|g_{\sigma\sigma}|}\right)$ (no sum) as required.

d) $$\Gamma^{\sigma}_{\sigma\sigma} = \partial_{\sigma}\left(\ln\sqrt{|g_{\sigma\sigma}|}\right).$$

We have: $\Gamma^{\sigma}_{\sigma\sigma} = \frac{1}{2}g^{\sigma\sigma}\frac{\partial g_{\sigma\sigma}}{\partial x^{\sigma}}\frac{1}{2}\left(g_{\sigma\sigma}\right)^{-1}\partial_{\sigma} g_{\sigma\sigma} = \partial_{\sigma}\left(\ln\sqrt{|g_{\sigma\sigma}|}\right)$ (no sum) as required.

2. A clock is in a circular orbit at $r = 10\ M$ in a Schwarzschild metric.

a) How much time elapses on the clock during one orbit? (Integrate the proper time $d\tau = |ds^2|^{1/2}$ over an orbit.)

We start with:

$$\left(\frac{dr}{d\tau}\right)^2 = c^2\left(\left(\frac{E}{mc^2}\right)^2 - 1\right) + \frac{2GM}{r} - \frac{L^2}{m^2r^2} + \frac{2GML^2}{m^2r^3c^2}, \quad \text{and}$$

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{L^2}{m^2r^3} - \frac{3GML^2}{m^2c^2r^4}.$$

We set $L/m = \tilde{L}$ and $E/mc^2 = \tilde{E}$. Then, $d^2r/d\tau^2 = 0$ at:

$$r_c = \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12\frac{G^2M^2}{c^2}\tilde{L}^2}}{2GM}, \quad \text{and} \quad \tilde{L}^2 = \frac{GMr_c}{1 - \frac{3GM}{r_cc^2}}.$$

Substituting $r_c = 10\ GM/c^2$, we find

$$\tilde{L} = \frac{10}{\sqrt{7}}\frac{GM}{c} \equiv r_c^2\frac{d\phi}{d\tau} \quad \Rightarrow \quad d\tau = 10\sqrt{7}\frac{GM}{c^3}d\phi \quad \text{for} \quad \theta = \frac{\pi}{2}.$$

Thus, $\Delta\tau = 20\pi\sqrt{7}GM/c^3$ for $\Delta\phi = 2\pi$.

b) It sends out a signal to a distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals?

For $dr/d\tau = 0$ we find:

$$\tilde{E}^2 = \left(1 - \frac{2GM}{r_cc^2}\right)\left(1 + \frac{\tilde{L}^2}{r_c^2c^2}\right) = \frac{\left(1 - \frac{2GM}{r_cc^2}\right)^2}{\left(1 - \frac{3GM}{r_cc^2}\right)}.$$

Hence:

$$\tilde{E} = \frac{\left(1 - \frac{2GM}{r_cc^2}\right)}{\sqrt{1 - \frac{3GM}{r_cc^2}}} \equiv \left(1 - \frac{2GM}{r_cc^2}\right)\frac{dt}{d\tau} \quad \Rightarrow \quad dt = \sqrt{\frac{10}{7}}d\tau.$$

Thus, $\Delta t = 20\pi\sqrt{10}GM/c^3$ for $\Delta\phi = 2\pi$.

c) A second clock is located at rest at $r = 10\ M$ next to the orbit of the first clock. (Rockets keep it there.) How much time elapses on it between successive passes of the orbiting clock?

For this clock, we have $d\tau_s = \sqrt{\frac{1-2GM}{r_cc^2}}dt \ \Rightarrow\ \Delta\tau_s = \sqrt{\frac{4}{5}}\Delta t = 40\pi\sqrt{2}GM/c^3$ for $\Delta\phi = 2\pi$.

d) Calculate (b) again in seconds for an orbit at $r = 6\ M$ where $M = 14M_\odot$. This is the minimum fluctuation time one expects in the X-ray spectrum of Cyg X-1: why?

From above, we have:

$$\tilde{L} = \frac{\sqrt{GMr_c}}{\sqrt{1 - \frac{3GM}{r_cc^2}}} = r_c^2\frac{d\phi}{d\tau} \quad \text{and} \quad \tilde{E} = \frac{\left(1 - \frac{2GM}{r_cc^2}\right)}{\sqrt{1 - \frac{3GM}{r_cc^2}}} = \left(1 - \frac{2GM}{r_cc^2}\right)\frac{dt}{d\tau} \quad \Rightarrow \quad \frac{d\phi}{dt} = \sqrt{\frac{GM}{r_c^3}},$$

exactly as in the Newtonian case. All circular orbits at $r_c < 6\ M$ are unstable and all orbits at $r_c > 6\ M$ have longer periods, so the last (marginally) stable orbit at $r = 6\ M$ gives the shortest timescale ($\Delta t = 12\pi\sqrt{6}GM/c^3$) for fluctuations. With $G = 6.67\times10^{-11}\,\mathrm{m^2/kg/s^3}$, $M_\odot = 1.99\times10^{30}\,\mathrm{kg}$ and $c = 3.00\times10^8\,\mathrm{m/s^2}$, and for $M = 14M_\odot$, this gives 6.4 ms.

e) If the orbiting "clock" is the twin Artemis, in the orbit in (d), how much does she age during the time her twin Diana lives 40 years far from the black hole and at rest with respect to it?

We find from part d) that $d\tau = \sqrt{1-\frac{3GM}{r_c c^2}}dt$, so $\Delta\tau_{\mathrm{A}} = \frac{1}{\sqrt{2}}\Delta t_{\mathrm{D}} = 20\sqrt{2}$ years.

3. Consider the Schwarzschild metric in the form::

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$$

a) Find the coordinate transformation $\hat{t}(t,r)$ such that it becomes :

$$ds^2 = -a(r)d\hat{t}^2 + 2b(r)d\hat{t}dr + dr^2 + r^2 d\Omega^2.$$

First we note that $1/(1-2M/r) = 1 + \frac{2M}{r-2M}$. Then we can realize that $a(r) = 1 - \frac{2M}{r}$ and that we must have $\hat{t} = t + f(r)$. Then differentiate and substitute (using $f'(r) = df/dr$):

$$-\left(1-\frac{2M}{r}\right)\left(dt + f'(r)dr\right)^2 + 2b(r)\left(dt + f'(r)dr\right)dr = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{2M}{r-2M}dr^2.$$

This gives $\left(1-\frac{2M}{r}\right)f'(r) = b(r)$ and $-\left(1-\frac{2M}{r}\right)f'(r)^2 + 2b(r)f'(r) = \frac{2M}{r-2M}$. Thus, we find:

$$b(r) = \sqrt{\frac{2M}{r}}, \quad \text{and} \quad f'(r) = \frac{b(r)}{\left(1-\frac{2M}{r}\right)} \Rightarrow f(r) = 2\sqrt{2Mr} + 2M\ln\left(\frac{\sqrt{r}-\sqrt{2M}}{\sqrt{r}+\sqrt{2M}}\right).$$

b) Show that, in these coordinates, the metric can also be written as:

$$ds^2 = -d\hat{t}^2 + \left(dr + b(r)d\hat{t}\right)^2 + r^2 d\Omega^2.$$

We have:

$$-d\hat{t}^2 + b(r)^2 d\hat{t}^2 + 2b(r)dtdr + dr^2 = -d\hat{t}^2 + \left(dr + b(r)d\hat{t}\right)^2,$$

as required.

4. A good approximation to the metric outside the surface of the Earth is provided by:

$$ds^2 = -(c^2 + 2\Phi)dt^2 + (1 - 2\Phi/c^2)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where

$$\Phi = -\frac{GM}{r}$$

may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the Earth. For this problem Φ may assumed to be small.

a) Imagine a clock on the surface of the Earth a distance R_1 from the Earth's center, and another clock on a tall building at a distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time, t. Which clock runs faster?

We will assume the Earth is not rotating. Rotation does have some effect, but the metric given corresponds to a non-rotating body. Also, we will work to first order in Φ throughout. A clock sitting on the surface of the Earth is not following a geodesic. In this case, we can say $\Delta\tau_1 = \sqrt{1 - \frac{2GM}{R_1 C^2}}\Delta t$ and $\Delta\tau_2 = \sqrt{1 - \frac{2GM}{R_2 C^2}}\Delta t \approx \Delta\tau_1 + \frac{gh}{c^2}\frac{(\Delta t)^2}{\Delta\tau_1} > \Delta\tau_1$, where $h = R_2 - R_1$. The clock on the tall building runs faster that the clock on the surface. *Note*: With $\sqrt{1 + 2\frac{\Phi}{c^2}} \approx 1 + \frac{\Phi}{c^2} \approx 1/\sqrt{1 - 2\frac{\Phi}{c^2}}$, we can effectively use the Schwarzschild metric now.

b) Solve for a geodesic around the center of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

For a circular orbit, we have from problem 2 d):

$$\tilde{L} = \frac{\sqrt{GMr_c}}{\sqrt{1 - \frac{3GM}{R_1 c^2}}} = R_1^2\frac{d\phi}{d\tau} \quad \text{and} \quad \tilde{E} = \frac{\left(1 - \frac{2GM}{R_1 c^2}\right)}{\sqrt{1 - \frac{3GM}{R_1 c^2}}} = \left(1 - \frac{2GM}{R_1 c^2}\right)\frac{dt}{d\tau} \quad \Rightarrow \quad \frac{d\phi}{dt} = \sqrt{\frac{GM}{R_1^3}}.$$

For $M = M_E = 5.972 \times 10^{24}\,\text{kg}$ and $R_1 = R_E = 6.378 \times 10^6\,\text{m}$, and using $G = 6.674 \times 10^{-11}\,\text{N(m/kg)}^2$, this gives $\frac{d\phi}{dt} = 1.239\,\text{mHz}$, corresponding to a period of 5069 s.

c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual numbers for the radius of the earth, and so on, to get an answer in seconds.

We have (using $c = 2.998 \times 10^8\,\text{m/s}^2$):

$$\Delta\tau_c = 2\pi\sqrt{\frac{R_1^3}{GM}}\sqrt{1 - \frac{3GM}{R_1 c^2}} = 5069\,\text{s} \times \left(1 - 1.043 \times 10^{-9}\right) = 5069\,\text{s} - 5.3\,\mu\text{s}.$$

d) How does this number compare with the proper time elapsed on the clock stationary on the surface of the Earth?

For the clock stationary on the Earth's surface, we have:

$$\Delta\tau_S = \sqrt{1-\frac{2GM}{R_1c^2}}\Delta t_C = 2\pi\sqrt{\frac{R_1^3}{GM}}\sqrt{1-\frac{2GM}{R_1c^2}} = 5069\left(1-6.953\times10^{-10}\right)\mathrm{s} = 5069\,\mathrm{s} - 3.5\,\mu\mathrm{s}.$$

We see that the orbiting clock ticks slightly slower than the stationary clock.

5. Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM/c^2$. Use the ordinary Schwarzschoild coordinates (t, r, θ, ϕ).

a) Show that the radial coordinate must decrease at a minimum rate given by:

$$\left|\frac{dr}{d\tau}\right| = \sqrt{\frac{2GM}{r} - c^2}.$$

We have:

$$\begin{aligned}\left(\frac{dr}{d\tau}\right)^2 &= c^2\left(\left(\frac{E}{mc^2}\right)^2 - 1\right) + \frac{2GM}{r} - \frac{L^2}{m^2r^2} + \frac{2GML^2}{m^2r^3c^2},\\ &= \frac{2GM}{r} - c^2 + \frac{L^2}{m^2r^2}\left(\frac{2GM}{r} - 1\right) + \left(\frac{E}{mc}\right)^2 \geq \frac{2GM}{r} - c^2 \text{ for } r \leq 2M.\end{aligned}$$

b) Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM/c^2$ to $r = 0$.

Integrating, we find

$$\Delta\tau \leq -\frac{1}{c}\int_{2GM/c^2}^{0}\frac{dr}{\sqrt{\frac{2GM}{rc^2} - 1}} = \frac{2GM}{c^3}\int_0^{\pi/2}(1-\cos 2y)\,dy = \frac{\pi GM}{c^3},$$

where we have substituted: $r = 2GM/c^2 \times \sin^2 y$.

c) Express this in seconds for a black hole with a mass measured in Solar masses, $M_\odot$.

Using $G = 6.67\times10^{-11}\,\mathrm{m^2/kg/s^3}$, $M_\odot = 1.99\times10^{30}\,\mathrm{kg}$ and $c = 3.00\times10^8\,\mathrm{m/s^2}$, we find $\Delta\tau \leq \frac{M}{M_\odot}15.4\,\mu\mathrm{s}$

d) Show that this maximum proper time is achieved by falling freely with $E \to 0$.

From part a) we see that $dr/d\tau = \sqrt{\frac{2GM}{r} - c^2}$ (giving equality in the result above), iff $\tilde{L} = 0$ (corresponding to free fall with no angular momentum), and $\tilde{E} = 0$ (which cannot happen from outside the black hole).

Note: $\tilde{E} = (1 - \frac{2GM}{rc^2})\frac{dt}{d\tau} \neq 0$ unless $t =$ constant, but would have $dt < 0$. Why?