

PHZ 6607, Special and General Relativity I

Class Number 21787, Fall 2018, Homework 3

Due at the start of class on Friday, November 9.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. Imagine that we have a *diagonal* metric, $g_{\mu\nu}$. Show that the Christoffel symbols are given by (*Note*: In these equations, $\mu \neq \nu \neq \sigma$, and repeated indices are **NOT** summed):

a) $\Gamma_{\mu\nu}^{\sigma} = 0,$

b) $\Gamma_{\mu\mu}^{\sigma} = -\frac{1}{2}(g_{\sigma\sigma})^{-1}\partial_{\sigma}g_{\mu\mu},$

c) $\Gamma_{\mu\sigma}^{\sigma} = \partial_{\mu} \left(\ln \sqrt{|g_{\sigma\sigma}|} \right),$

d) $\Gamma_{\sigma\sigma}^{\sigma} = \partial_{\sigma} \left(\ln \sqrt{|g_{\sigma\sigma}|} \right).$

2. A clock is in a circular orbit at $r = 10 M$ in a Schwarzschild metric.

- a) How much time elapses on the clock during one orbit? (Integrate the proper time $d\tau = |ds^2|^{1/2}$ over an orbit.)
- b) It sends out a signal to a distant observer once each orbit. What time interval does the distant observer measure between receiving any two signals?
- c) A second clock is located at rest at $r = 10 M$ next to the orbit of the first clock. (Rockets keep it there.) How much time elapses on it between successive passes of the orbiting clock?
- d) Calculate (b) again in seconds for an orbit at $r = 6 M$ where $M = 14M_{\odot}$. This is the minimum fluctuation time one expects in the X-ray spectrum of Cyg X-1: why?
- e) If the orbiting “clock” is the twin Artemis, in the orbit in (d), how much does she age during the time her twin Diana lives 40 years far from the black hole and at rest with respect to it?

3. Consider the Schwarzschild metric in the form::

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2$$

- a) Find the coordinate transformation $\hat{t}(t, r)$ such that it becomes :

$$ds^2 = -a(r)d\hat{t}^2 + 2b(r)d\hat{t}dr + dr^2 + r^2d\Omega^2.$$

- b) Show that, in these coordinates, the metric can also be written as:

$$ds^2 = -d\hat{t}^2 + (dr + b(r)d\hat{t})^2 + r^2d\Omega^2.$$

4. A good approximation to the metric outside the surface of the Earth is provided by:

$$ds^2 = -(c^2 + 2\Phi)dt^2 + (1 - 2\Phi/c^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$\Phi = -\frac{2GM}{r}$$

may be thought of as the familiar Newtonian gravitational potential. Here G is Newton's constant and M is the mass of the Earth. For this problem Φ may assumed to be small.

- a) Imagine a clock on the surface of the Earth a distance R_1 from the Earth's center, and another clock on a tall building at a distance R_2 from the Earth's center. Calculate the time elapsed on each clock as a function of the coordinate time, t . Which clock runs faster?
- b) Solve for a geodesic around the center of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?
- c) How much proper time elapses while a satellite at radius R_1 (skimming along the surface of the Earth, neglecting air resistance) completes one orbit? You can work to first order in Φ if you like. Plug in the actual numbers for the radius of the earth, and so on, to get an answer in seconds.
- d) How does this number compare with the proper time elapsed on the clock stationary on the surface of the Earth?
5. Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM/c^2$. Use the ordinary Schwarzschild coordinates (t, r, θ, ϕ) .

- a) Show that the radial coordinate must decrease at a minimum rate given by:

$$\left| \frac{dr}{d\tau} \right| = \sqrt{\frac{2GM}{r} - c^2}.$$

- b) Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM/c^2$ to $r = 0$.
- c) Express this in seconds for a black hole with a mass measured in Solar masses, M_\odot .
- d) Show that this maximum proper time is achieved by falling freely with $E \rightarrow 0$.