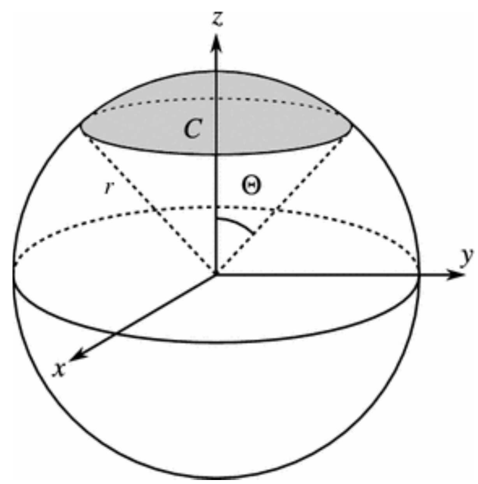
PHZ 6607, Special and General Relativity I

Class Number 21787, Fall 2018, Homework 2

Due at the start of class on Friday, October 12.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. This question concerns a measure of curvature known as the *Gaussian Curvature.*



a) For a convex, n-sided polygon on a flat two dimensional plane, what is the sum of all the exterior angles? Subtract this total from 2π. The result tells you something about the curvature of the 2-plane.

For a triangle, the interior and exterior angles add to 3π, and the interior angles add to π, so the difference is 2π. For a square, The interior and exterior angles add to 4π, and the interior angles add to 2π, so the difference is again 2π. For an n-sided polygon, the result is always the same: 2π. Subtracting this from 2π we get 0. Any other result would have indicated curvature.

b) Find the area, A_C, of the spherical cap, C, shown in the diagram.

The area if the spherical cap is given by

$$A_C = r^2 \int_0^{2\pi} \int_0^{\Theta} \sin\theta d\theta d\phi = 2\pi r^2 (1 - \cos\Theta) = 4\pi r^2 \sin^2(\Theta/2).$$

Note: this has the sensible limit $\pi(r\Theta)^2$ as $\Theta \to 0$, and $4\pi r^2$ as $\Theta \to \pi$.

c) On the parallel of latitude which bounds C, parallel transport the initial vector $\mathbf{V} = (V_0^\theta, V_0^\phi)$ from $\phi = 0$ to $\phi = 2\pi$, using ϕ as the parameter along the (non-geodesic) curve.

From class notes (9/19/18, where we have replaced θ_0 by Θ), we have:

$$V^\theta(\phi) = V_0^\theta \cos(\phi\cos\Theta) + \sin\Theta V_0^\phi \sin(\phi\cos\Theta),$$
$$V^\phi(\phi) = V_0^\phi \cos(\phi\cos\Theta) - \frac{1}{\sin\Theta} V_0^\theta \sin(\phi\cos\Theta).$$

Substituting $\phi = 2\pi$ we obtain:

$$V^\theta(2\pi) = V_0^\theta \cos(2\pi\cos\Theta) + \sin\Theta V_0^\phi \sin(2\pi\cos\Theta),$$
$$V^\phi(2\pi) = V_0^\phi \cos(2\pi\cos\Theta) - \frac{1}{\sin\Theta} V_0^\theta \sin(2\pi\cos\Theta).$$

d) Show that the magnitude of the vector, $\mathbf{V}$, has not changed, and that the vector has rotated through an angle $\Delta = 2\pi\cos\Theta$ with respect to its original direction.

For $|\mathbf{V}(\phi)|^2 = V^\mu V^\nu g_{\mu\nu}$ we obtain $r^2\left((V^\theta)^2 + \sin^2\theta_0 (V^\phi)^2\right)$, independent of ϕ. The angle of rotation is given by:

$$\Delta = \arccos\left(\frac{\mathbf{V}(0)\cdot\mathbf{V}(2\pi)}{|\mathbf{V}(0)|\times|\mathbf{V}(2\pi)|}\right) = \arccos(\cos(2\pi\cos\Theta)) = 2\pi\cos\Theta.$$

Thus, we have $2\pi - \Delta = 2\pi(1-\cos\Theta) = 4\pi\sin^2(\Theta/2)$.

e) Evaluate

$$\kappa_G = \frac{(2\pi - \Delta)}{A_C},$$

and show that it is a constant, independent of Θ. The quantity, κ_G, is known as the Gaussian Curvature, and is constant for the sphere, as one might expect.

Substituting, we find

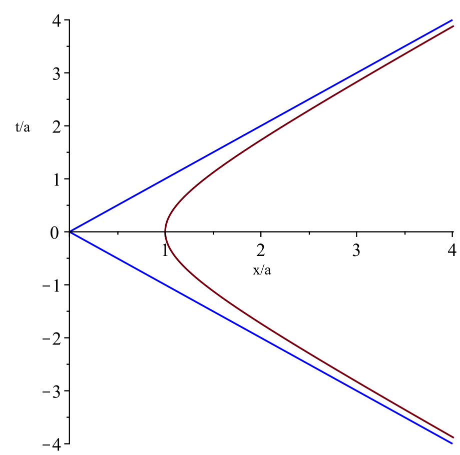
$$\kappa_G = \frac{4\pi\sin^2(\Theta/2)}{4\pi r^2\sin^2(\Theta/2)} = \frac{1}{r^2},$$

which is indeed independent of Θ.

2. The world line of a particle in flat Minkowski space is described by the parametric equations:

$$t(\lambda) = a\sinh\left(\frac{\lambda}{a}\right), \qquad x(\lambda) = a\cosh\left(\frac{\lambda}{a}\right),$$

in some Lorentz frame, where λ is the parameter and a is a constant. *Note:* t and x have the same dimension, so you should use $c = 1$.



a) Describe the motion and use a 1-plus-1, space-time diagram to represent it.

The motion is hyperbolic, as shown in the figure (in red), and is given by $x^2 - t^2 = a^2$.

b) Compute the particle's four-velocity and acceleration components.

We find:

$$\frac{dt}{d\tau} = \frac{d\lambda}{d\tau}\cosh\left(\frac{\lambda}{a}\right),$$
$$\frac{dx}{d\tau} = \frac{d\lambda}{d\tau}\sinh\left(\frac{\lambda}{a}\right), \quad \text{giving}$$
$$V^\mu V^\nu g_{\mu\nu} = -\left(\frac{d\lambda}{d\tau}\right)^2 \quad\Rightarrow\quad \frac{d\lambda}{d\tau} = 1.$$

Then:

$$V^\mu = \begin{pmatrix} \cosh\left(\frac{\lambda}{a}\right) \\ \sinh\left(\frac{\lambda}{a}\right) \end{pmatrix}, \quad \text{and} \quad A^\mu = \frac{1}{a}\begin{pmatrix} \sinh\left(\frac{\lambda}{a}\right) \\ \cosh\left(\frac{\lambda}{a}\right) \end{pmatrix}.$$

c) Show that λ is proper time along the world line and that the acceleration is uniform.

We have $d\lambda/d\tau = 1$ from above, so λ is equivalent to proper time along the trajectory. Also, $|\mathbf{A}| = 1/a$ is constant and $\mathbf{A}\cdot\mathbf{V} = 0$, so acceleration is uniform.

d) Interpret a.

From the statement of the problem, $[a] = L$, and we see $|\mathbf{A}| = 1/a$, so a is the reciprocal of the acceleration. *Note*: $[\mathbf{A}] = L/T^2 \equiv 1/L$ (since t has the dimensions of length) as required.

3. Consider Killing's equation $\nabla_{(\mu}\xi_{\nu)} = 0$, in flat, 4-dimensional Minkowski space, with $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

a) Show, by explicit calculation, that:

$$\begin{aligned} \mathbf{T}^\mu &= (1,0,0,0), \\ \mathbf{X}^\mu &= (0.1,0,0), \\ \mathbf{Y}^\mu &= (0,0,1,0), \\ \mathbf{Z}^\mu &= (0,0,0,1), \end{aligned}$$

each satisfy Killing's equation in flat space.

Note that $T_\mu = (-1,0,0,0)$, while $X_\mu = (0,1,0,0)$, $Y_\mu = (0,0,1,0)$, and $Z_\mu = (0,0,0,1)$, and that in cartesian coordinates in flat space the covariant derivative (∇) is given by the ordinary partial derivative (∂). Now, for $\nabla_{(\mu}T_{\nu)}$, we must have one of μ or ν equal to t, and the other one can be t or i. Both $\{\mu,\nu\} = \{t,t\}$ and $\{\mu,\nu\} = \{t,i\}$ give zero, since the non-zero component is constant. A similar argument applies for each of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$.

b) Show, by explicit calculation, that:

$$\begin{aligned} \mathbf{L}_x^\mu &= (0,0,-z,y), \\ \mathbf{L}_y^\mu &= (0,z,0,-x), \\ \mathbf{L}_z^\mu &= (0,-y,x,0), \end{aligned}$$

each satisfy Killing's equation in flat space.

We first compute $L^x_\mu = (0,0,-z,y)$, $L^y_\mu = (0,z,0,-x)$ and $L^z_\mu = (0,-y,x,0)$. Now, for $\nabla_{(\mu}L^x{}_{\nu)}$, we must have only y and/or z for μ and ν (because t and x must give zero). Then $\nabla_y L^x{}_y = 0 = \nabla_z L^x{}_z$, and $\nabla_{(y}L^x{}_{z)} = \partial_y L^x{}_z + \partial_z L^x{}_y = 1 - 1 = 0$, so Killing's equation is satisfied. Similarly, for $\nabla_{(\mu}L^y{}_{\nu)}$ only x and/or z can contribute for μ and ν, and $\nabla_x L^y{}_x = 0 = \nabla_z L^y{}_z$, while $\nabla_{(x}L^y{}_{z)} = \partial_x L^y{}_z + \partial_z L^y{}_x = -1+1 = 0$ too, so Killing's equation is again satisfied. That $\nabla_{(\mu}L^z{}_{\nu)} = 0$ similarly follows by cyclic permutation of the indices.

c) Show, by explicit calculation, that:

$$
\begin{aligned}
\mathbf{B}^\mu_x &= (x, t, 0, 0),\\
\mathbf{B}^\mu_y &= (y, 0, t, 0),\\
\mathbf{B}^\mu_z &= (z, 0, 0, t),
\end{aligned}
$$

each satisfy Killing's equation in flat space.

First we must compute $B^x{}_\mu = (-x, t, 0, 0)$, $B^y{}_\mu = (-y, 0, t, 0)$ and $B^z{}_\mu = (-z, 0, 0, t)$, then the rest of the computation proceeds exactly in the same way as for angular momentum.

d) Show that:

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$$

where $\mathcal{L}$ is the Lie derivative. First write the Lie derivative in terms of partial derivatives, and then convert these into covariant derivatives for the final result.

By definition:

$$
\begin{aligned}
\mathcal{L}_\xi g_{\mu\nu} &= \xi^\tau \partial_\tau g_{\mu\nu} + g_{\tau\nu}\partial_\mu \xi^\tau + g_{\mu\tau}\partial_\nu \xi^\tau\\
&= \xi^\tau \partial_\tau g_{\mu\nu} - \Gamma^\sigma_{\mu\tau} g_{\sigma\nu}\xi^\tau - \Gamma^\sigma_{\nu\tau} g_{\mu\sigma}\xi^\tau\\
&\quad + g_{\tau\nu}\partial_\mu \xi^\tau + g_{\tau\nu}\Gamma^\tau_{\sigma\mu}\xi^\sigma + g_{\mu\tau}\partial_\nu \xi^\tau + g_{\mu\tau}\Gamma^\tau_{\sigma\nu}\xi^\sigma\\
&= \xi^\tau \nabla_\tau g_{\mu\nu} + g_{\tau\nu}\nabla_\mu \xi^\tau + g_{\mu\tau}\nabla_\nu \xi^\tau\\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \xi^\tau\left(\nabla_\mu g_{\tau\nu} + \nabla_\nu g_{\mu\tau} - \nabla_\tau g_{\mu\nu}\right)\\
&= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu
\end{aligned}
$$

Where in the second equality we have used the symmetry of the connection, and in the last line we have used that the connection is metric compatible.

e) You have just shown that, for each Killing vector, the Lie derivative of the metric is zero along each Killing vector. What does this tell you?

Thus the metric remains unchanged under Lie transport in the direction of each Killing vector. In particular the Minkowski metric is unchanged by the action of each of these 10 vectors, which together generate translations, rotations and boosts.

4. Consider the Lagrangian density for Electromagnetism (in flat, Minkowski space-time):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, and is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{pmatrix}, \quad \text{and} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B_3 & -B_2 \\ -E^2 & -B_3 & 0 & B_1 \\ -E^3 & B_2 & -B_1 & 0 \end{pmatrix}$$

(note the placement of indices), J^μ is the current density 4-vector, and A_μ is the 4-vector potential.

a) Find the Euler-Lagrange equations of motion for A_μ.

First note that:

$$F_{\mu\nu}F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)F^{\mu\nu} = \partial_\mu A_\nu F^{\mu\nu} - \partial_\nu A_\mu F^{\mu\nu} = \partial_\mu A_\nu F^{\mu\nu} + \partial_\nu A_\mu F^{\nu\mu} = 2\partial_\mu A_\nu F^{\mu\nu}.$$

Then we find:

$$\partial_\mu F^{\mu\nu} + j^\nu = 0,$$

where, for $\nu = t$ we have

$$-\partial_i E^i + \rho = 0,$$

(in which we have used $J^0 = \rho$), and for $\nu = i$ we have

$$\dot{E}^i - \varepsilon^{ijk}\partial_j B_k + J^i = 0.$$

b) How many equations does this give you? What other equations do you need to add to obtain all of Maxwell's equations? Where do they come from?

We have four Maxwell's equations, and are missing the four without source terms, so we need to find them. They come from:

$$\partial_\mu F_{\nu\tau} + \partial_\mu F_{\nu\tau} + \partial_\mu F_{\nu\tau} = 0 \quad \Rightarrow \quad \tilde{\varepsilon}^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0,$$

which follows trivially from the definition of $F_{\mu\nu}$ in terms of A_μ, and the commutativity of partial derivatives. The $\mu = t$ equation gives:

$$\partial_i B^i = 0,$$

while the $\mu = i$ equation gives:

$$\varepsilon^{ijk}\partial_j E_k + \dot{B}^i = 0.$$

c) Express $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

First note that there are $4! = 24$ terms in this expression, but that:

$$\begin{aligned}\tilde{\epsilon}_{0123}F^{01}F^{23} &= \tilde{\epsilon}_{0132}F^{01}F^{32} = \tilde{\epsilon}_{1023}F^{10}F^{23} = \tilde{\epsilon}_{1032}F^{10}F^{32} \\ &= \tilde{\epsilon}_{2301}F^{23}F^{01} = \tilde{\epsilon}_{3201}F^{32}F^{01} = \tilde{\epsilon}_{2310}F^{23}F^{10} = \tilde{\epsilon}_{3210}F^{32}F^{10},\end{aligned}$$

so there are really only three independent terms. These can be written as:

$$\tilde{\epsilon}_{0123}F^{01}F^{23} + \tilde{\epsilon}_{0231}F^{02}F^{31} + \tilde{\epsilon}_{0312}F^{03}F^{12} = E^1B_1 + E^2B_2 + E^3B_3 = E^iB_i = \mathbf{E}\cdot\mathbf{B}.$$

d) Show that including $\mathcal{L}'$ as an addition to the Lagrangian density used above does not affect the result for Maxwell's equations.

Varying this term with respect to A_μ in the action would give us $4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$, and we have already used that this equals zero in part b) above. Let's see how it really comes about:

$$\begin{aligned}4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) &= 4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma - 4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\sigma A_\rho \\ &= 4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma - 4\,\tilde{\epsilon}^{\mu\nu\sigma\rho}\partial_\nu\partial_\rho A_\sigma \\ &= 4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma + 4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma \\ &= 8\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma,\end{aligned}$$

which always vanishes because $\partial_\nu\partial_\rho A_\sigma$ is symmetric in (ν,ρ) while $\tilde{\epsilon}^{\mu\nu\rho\sigma}$ is antisymmetric.

e) Give a deep reason that could explain why this would be the case?

A term in the action which contributes nothing to the action must be equivalent to a total derivative. We can see how this comes about here:

$$\tilde{\epsilon}^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -4\,\tilde{\epsilon}^{\mu\nu\rho\sigma}\left(\partial_\nu A_\mu\right)\left(\partial_\rho A_\sigma\right) = -4\partial_\nu\left(\tilde{\epsilon}^{\mu\nu\rho\sigma}A_\mu\partial_\rho A_\sigma\right) + 4A_\mu\left(\tilde{\epsilon}^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma\right),$$

where the first term **is** a total derivative, and the second term vanishes by symmetry.
Note: ∂ and $\tilde{\epsilon}$ are deliberately used here, rather that ∇ and ϵ. Why?

5. Let K^μ be a Killing vector (*i.e.*, $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$ is satisfied).

a) Using $[\nabla_\mu,\nabla_\nu]K_\rho = R_\rho{}^\sigma{}_{\mu\nu}K_\sigma$, along with $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$, and the symmetries of the Riemann tensor, show that $\nabla_\mu\nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu}K^\nu$.

We commute the derivatives and use Killing's equation several times, and then use symmetries of the Riemann tensor:

$$\begin{aligned}\nabla_\mu\nabla_\sigma K_\rho &= \nabla_\sigma\nabla_\mu K_\rho + R_\rho{}^\nu{}_{\mu\sigma}K_\nu = -\nabla_\sigma\nabla_\rho K_\mu + R_\rho{}^\nu{}_{\mu\sigma}K_\nu \\ &= -\nabla_\rho\nabla_\sigma K_\mu - R_\mu{}^\sigma{}_{\sigma\rho}K_\sigma + R_\rho{}^\nu{}_{\mu\sigma}K_\nu \\ &= \nabla_\rho\nabla_\mu K_\sigma - R_\mu{}^\nu{}_{\sigma\rho}K_\nu + R_\rho{}^\nu{}_{\mu\sigma}K_\nu \\ &= \nabla_\mu\nabla_\rho K_\sigma + R_\sigma{}^\nu{}_{\rho\mu}K_\nu - R_\mu{}^\nu{}_{\sigma\rho}K_\nu + R_\rho{}^\nu{}_{\mu\sigma}K_\nu \\ &= -\nabla_\mu\nabla_\sigma K_\rho + R_\sigma{}^\nu{}_{\rho\mu}K_\nu - R_\mu{}^\nu{}_{\sigma\rho}K_\nu + R_\rho{}^\nu{}_{\mu\sigma}K_\nu \\ &= -\nabla_\mu\nabla_\sigma K_\rho - 2R_\mu{}^\nu{}_{\sigma\rho}K_\nu \\ &= -\nabla_\mu\nabla_\sigma K_\rho + 2R_{\rho\sigma\mu}{}^\nu K_\nu,\end{aligned}$$

which gives the desired result, $\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$, after some further index juggling.

b) By contraction to obtain $\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu$, one further differentiation, and additional manipulation, show that $K^\lambda \nabla_\lambda R = 0$, *i.e.*, that the Ricci scalar does not change along a Killing vector field.

Contraction indeed gives $\nabla_\mu \nabla_\sigma K^\mu = R^\mu{}_{\sigma\mu\nu} K^\nu = R_{\sigma\nu} K^\nu$. Then we take one extra derivative of this equation:

$$\nabla^\sigma \nabla_\mu \nabla_\sigma K^\mu = \nabla^\sigma R_{\sigma\nu} K^\nu + R_{\sigma\nu} \nabla^\sigma K^\nu = \frac{1}{2} \nabla_\nu R K^\nu = \frac{1}{2} K^\nu \nabla_\nu R,$$

where the second term vanishes since $R_{\sigma\nu}$ is symmetric and $\nabla^\sigma K^\nu$ is antisymmetric in (σ, ν). Alternatively, by commuting the outer derivatives and using Killing's equation, we find:

$$\begin{aligned}
\nabla^\sigma \nabla_\mu \nabla_\sigma K^\mu &= \nabla_\mu \nabla^\sigma \nabla_\sigma K^\mu + R^\mu{}_\nu{}^\sigma{}_\mu \nabla_\sigma K^\nu + R_\sigma{}^{\nu\sigma}{}_\mu \nabla_\nu K^\mu \\
&= \nabla_\mu \nabla^\sigma \nabla_\sigma K^\mu - R_\nu{}^\sigma \nabla_\sigma K^\nu + R^\nu{}_\mu \nabla_\nu K^\mu \\
&= \nabla_\mu \nabla^\sigma \nabla_\sigma K^\mu, \quad \text{since the Ricci terms vanish by symmetry} \\
&= -\nabla_\mu \nabla^\sigma \nabla^\mu K_\sigma \\
&= -\nabla^\sigma \nabla_\mu \nabla_\sigma K^\mu = 0,
\end{aligned}$$

(where, in the last line we used $\sigma \leftrightarrow \mu$ and changed the position of all indices – up and down), where the final result holds since the last line and the left hand side are equal and opposite. Thus, $K^\nu \nabla_\nu R = 0$ as required.