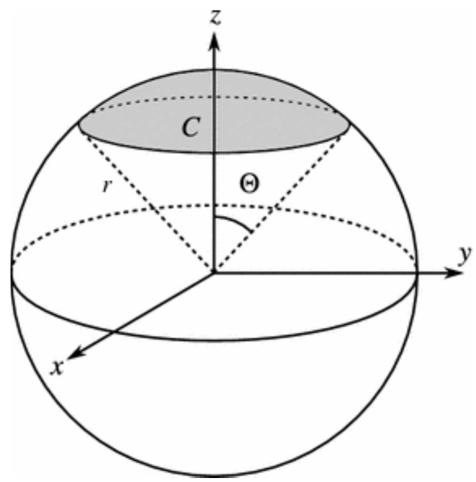
PHZ 6607, Special and General Relativity I

Class Number 21787, Fall 2018, Homework 2

Due at the start of class on Friday, October 12.

Answer all questions. Please write neatly and include your name on the front page of your answers. To gain maximum credit you should explain your reasoning and show all working.

1. This question concerns a measure of curvature known as the *Gaussian Curvature.*



a) For a convex, n-sided polygon on a flat two dimensional plane, what is the sum of all the exterior angles? Subtract this total from 2π. The result tells you something about the curvature of the 2-plane.

b) Find the area, A_C, of the spherical cap, C, shown in the diagram.

c) On the parallel of latitude which bounds C, parallel transport the initial vector $\mathbf{V} = (V_0^\theta, V_0^\phi)$ from $\phi = 0$ to $\phi = 2\pi$, using ϕ as the parameter along the (non-geodesic) curve.

d) Show that the magnitude of the vector, $\mathbf{V}$, has not changed, and that the vector has rotated through an angle $\Delta = 2\pi \cos\Theta$ with respect to its original direction.

e) Evaluate

$$\kappa_G = \frac{(2\pi - \Delta)}{A_C},$$

and show that it is a constant, independent of Θ. The quantity, κ_G, is known as the Gaussian Curvature, and is constant for the sphere, as one might expect.

2. The world line of a particle in flat Minkowski space is described by the parametric equations in some Lorentz frame

$$t(\lambda) = a \sinh\left(\frac{\lambda}{a}\right), \qquad x(\lambda) = a \cosh\left(\frac{\lambda}{a}\right),$$

where λ is the parameter and a is a constant. *Note:* t and x have the same dimension, so you should use $c = 1$.

a) Describe the motion and use a 1-plus-1, space-time diagram to represent it.

b) Compute the particle's four-velocity and acceleration components.

c) Show that λ is proper time along the world line and that the acceleration is uniform.

d) Interpret a.

3. Consider Killing's equation $\nabla_{(\tau}\xi_{\nu)} = 0$, in flat, 4-dimensional Minkowski space, with $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

a) Show, by explicit calculation, that:

$$\begin{aligned} \mathbf{T}^{\mu} &= (1, 0, 0, 0), \\ \mathbf{X}^{\mu} &= (0.1, 0, 0), \\ \mathbf{Y}^{\mu} &= (0, 0, 1, 0), \\ \mathbf{Z}^{\mu} &= (0, 0, 0, 1), \end{aligned}$$

each satisfy Killing's equation in flat space.

b) Show, by explicit calculation, that:

$$\begin{aligned} \mathbf{L}_x^{\mu} &= (0, 0, -z, y), \\ \mathbf{L}_y^{\mu} &= (0, z, 0, -x), \\ \mathbf{L}_z^{\mu} &= (0, -y, x, 0), \end{aligned}$$

each satisfy Killing's equation in flat space.

c) Show, by explicit calculation, that:

$$\begin{aligned} \mathbf{B}_x^{\mu} &= (x, t, 0, 0), \\ \mathbf{B}_y^{\mu} &= (y, 0, t, 0), \\ \mathbf{B}_z^{\mu} &= (z, 0, 0, t), \end{aligned}$$

each satisfy Killing's equation in flat space.

d) Show that:

$$\mathcal{L}_{\xi} g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$$

where $\mathcal{L}$ is the Lie derivative. First write the Lie derivative in terms of partial derivatives, and then convert these into covariant derivatives for the final result.

e) You have just shown that, for each Killing vector, the Lie derivative of the metric is zero along each Killing vector. What does this tell you?

4. Consider the Lagrangian density for Electromagnetism (in flat, Minkowski space-time):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_{\mu} J^{\mu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, and is given by:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B^3 & -B^2 \\ E_2 & -B^3 & 0 & B^1 \\ E_3 & B^2 & -B^1 & 0 \end{pmatrix}, \quad \text{and} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B_3 & -B_2 \\ -E^2 & -B_3 & 0 & B_1 \\ -E^3 & B_2 & -B_1 & 0 \end{pmatrix}$$

(note the placement of indices), J^μ is the current density 4-vector, and A_μ is the 4-vector potential.

a) Find the Euler-Lagrange equations of motion for A_μ.

b) How many equations does this give you? What other equations do you need to add to obtain all of Maxwell's equations? Where do they come from?

c) Express $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

d) Show that including $\mathcal{L}'$ as an addition to the Lagrangian density used above does not affect the result for Maxwell's equations.

e) Give a deep reason that could explain why this would be the case?

5. Let K^μ be a Killing vector (*i.e.*, $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$ is satisfied).

a) Using $[\nabla_\mu, \nabla_\nu] K_\rho = R_\rho{}^\sigma{}_{\mu\nu} K_\sigma$, along with $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$, and the symmetries of the Riemann tensor, show that $\nabla_\mu \nabla_\sigma K^\rho = R^\rho{}_{\sigma\mu\nu} K^\nu$.

b) By contraction to obtain $\nabla_\mu \nabla_\sigma K^\mu = R_{\sigma\nu} K^\nu$, one further differentiation, and additional manipulation, show that $K^\lambda \nabla_\lambda R = 0$, *i.e.*, that the Ricci scalar does not change along a Killing vector field.