Questions for PHZ 6607, Special and General Relativity I

1) Consider an arbitrary point (t, x, y, z) and a nearby point $(t + \Delta t, x + \Delta x, y, z)$.

a) Write down an expression for the invariant separation between the two points (to second order): indicate how to tell when the separation is spacelike and when it is timelike;

Now consider a description of the two points with respect to a frame moving in the x-direction with speed v.

b) Find expressions for $\Delta t'$ and $\Delta x'$, the coordinate difference between the two points in the moving frame, and repeat part a) in terms of these primed quantities; and

c) On the basis of your answers above, comment on an implied property of the Lorentz transformation.

2a) Write out in full the non-relativistic equations for the motion of a free particle in spherical polar coordinates.

b) Read off the connection coefficients which underlie the identification of these equations of motion as the equations for a geodesic parameterized by Newtonian time, and explain exactly how you do this.

c) Find as many independent constants of the motion as you can (and compare with the situation in quantum mechanics).

3a) Show for a differentiable function φ defined over a manifold that the set $\{\partial\varphi/\partial x^a\}$ transforms as do the components of a tensor of rank one; what type of tensor has this construction defined?

b) Write out the components of this tensor in polar coordinates explicitly in terms of its components in cartesian coordinates.

c) Compare this with the expression you have previously used for the gradient of a function in spherical polar coordinates, and comment on any differences you see.

4) Maxwell's equations for electromagnetism may be written in a geometrically covariant form as:

$$\nabla_i B^i = 0,\ \nabla_i E^i = 4\pi\rho,\ \varepsilon^{ijk}\nabla_j E_k = -\dot{B}^i,\ \varepsilon^{ijk}\nabla_j B_k = \dot{E}^i + 4\pi J^i,$$

where ∇_i is a derivative operator (generally not $\partial/\partial x^i$), and E^i is not to be confused with E_i. In accord with these equations, the *electromagnetic stress tensor* T_i^j may be defined according to the formula:

$$4\pi T_i^j = -E_i E^j - B_i B^j + \frac{1}{2}\delta_i^j(E_k E^k + B_k B^k).$$

a) Try to show that, in a vacuum, it follows as a consequence of Maxwell's equations that

$$\dot{P}_i + \nabla_j T_i^j = 0,$$

where P_i is the electromagnetic energy flux vector defined by

$$4\pi P_i = \varepsilon_{ijk} E^j B^k.$$

Indicate as precisely as you can what extra information you would need to complete the proof.

b) Suppose there exists a symmetric second rank tensor g_{ij} which allows you to relate the covector E_i to the contravariant vector E^i via $E_i = g_{ij}E^j$. If this is to be enough extra information to prove the above conservation equations, what can you deduce about $\nabla_k g_{ij}$?

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5a) Using the Lagrangian

$$\pounds = \frac{1}{2} m g_{\mu\nu} \dot{x^\mu} \dot{x^\nu},$$

in cartesian (spatial) coordinates, find expressions for the conserved linear and angular canonical momenta p_x, p_y, p_z, L_x, L_y and L_z, in terms of the 'velocities'.

b) Similarly, starting with the Lagrangian written in spherical polar coordinates, find expressions for the canonical momenta p_r, p_θ and p_ϕ in terms of the 'velocities' which arise for these coordinates.

c) Use the coordinate relations:

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \sin\phi \text{ and},$$
$$z = r \cos\theta$$

write down the relations giving the cartesian velocities in terms of the polar velocities.

d) Hence, obtain expressions for all the conserved momenta listed in part a) in terms of the canonical momenta for polar coordinates.

e) What type of tensor are the velocities and what type the momenta? (This difference explains why it is dangerous to go directly from the velocities in one set of coordinates to the momenta in another. It also explains why $\mathbf{p} \cdot \mathbf{x}$ is always a scalar.)

6) Consider the four-dimensional spacetime metric which in $\{t, r, \theta, \phi\}$ coordinates has diagonal components $\{-c^2(1 - r_s/r), 1/(1 - r_s/r), r^2, r^2 \sin^2\theta\}$, all other components being zero. Notice that in the usual sense, the parameter r_s has the dimensions of length.

a) Write out expressions for the canonical momenta of a particle whose Lagrangian is given by:

$$\pounds = \frac{1}{2} m g_{\mu\nu} \dot{x^\mu} \dot{x^\nu}, \text{ in which } \dot{} = d/d\lambda,$$

in which the spacetime geometry is given by this metric.

b) Write out Hamilton's equations of motion for this particle, and indicate which canonical momenta are conserved.

c) By explicit calculation, show that all components of the angular momentum which were defined in Question 5) are in fact conserved for the particle moving in this non-flat geometry. Hence, or otherwize, argue whether or not this spacetime is spherically symmetric.

d) Find an expression for $dr/d\phi$, assuming the motion takes place in the equatorial plane. Obtain the corresponding equation for a particle of mass m moving under Newtonian gravity around a central body of mass M.

e) Using your knowledge of the difference between Newtonian and relativistic physics, and by assuming that the motion takes place in a region where both $|\mathbf{v}|/c$ and the ratio r_s/r are small, indicate how the resulting motion in this spacetime can be compared with motion in a Newtonian gravitational potential. Find the implied correspondence between the parameter r_s and the mass causing the potential. If you have used $c = 1$, restore it in the final relation you obtain.