Questions for PHZ 6607, Special and General Relativity I

1) Consider an arbitrary point (t, x, y, z) and a nearby point $(t + \Delta t, x + \Delta x, y, z)$.

a) Write down an expression for the invariant separation between the two points (to second order): indicate how to tell when the separation is spacelike and when it is timelike;

We find:

$$\Delta l^2 = -c^2 \Delta t^2 + \Delta x^2 \begin{cases} > 0 & \text{spacelike,} \\ = 0 & \text{lightlike,} \\ < 0 & \text{timelike.} \end{cases}$$

Now consider a description of the two points with respect to a frame moving in the x-direction with speed v.

b) Find expressions for $\Delta t'$ and $\Delta x'$, the coordinate difference between the two points in the moving frame, and repeat part a) in terms of these primed quantities; and

We have:

$$\begin{aligned} \Delta t' &= \gamma(\Delta t - v \Delta x / c^2) \\ \Delta x' &= \gamma(\Delta x - v \Delta t), \quad \text{where} \quad \gamma = 1/\sqrt{1 - v^2/c^2}. \end{aligned}$$

Thus:

$$\begin{aligned} \Delta l'^2 &= -c^2 \Delta t'^2 + \Delta x'^2 \\ &= -c^2 \gamma^2 \left(\Delta t^2 - \frac{2v \Delta x \Delta t}{c^2} + \frac{v^2 \Delta x^2}{c^4} \right) + \gamma^2 \left(\Delta x^2 - 2v \Delta x \Delta t + v^2 \Delta t^2 \right) \\ &= \gamma^2 \left(1 - \frac{v^2}{c^2} \right) \left(-c^2 \Delta t^2 + \Delta x^2 \right) \\ &= -c^2 \Delta t^2 + \Delta x^2 = \Delta l^2. \end{aligned}$$

c) On the basis of your answers above, comment on an implied property of the Lorentz transformation.

Lorentz transformations preserve Δl^2, whether it is spacelike, lightlike or timelike.

2a) Write out in full the non-relativistic equations for the motion of a free particle in spherical polar coordinates.

We have:

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right).$$

Hence:

$$\begin{aligned} \ddot{r} - r \dot{\theta}^2 - r \sin^2\theta \, \dot{\phi}^2 &= 0, \\ r^2 \ddot{\theta} + 2r \dot{r} \dot{\theta} - r \sin\theta \cos\theta \, \dot{\phi}^2 &= 0, \quad \text{and} \\ r^2 \sin^2\theta \, \ddot{\phi} + 2r \sin^2\theta \, \dot{r} \dot{\phi} + 2r^2 \sin\theta \cos\theta \, \dot{\theta} \dot{\phi} &= 0. \end{aligned}$$

b) Read off the connection coefficients which underlie the identification of these equations of motion as the equations for a geodesic parameterized by Newtonian time, and explain exactly how you do this.

From these we can identify and read off:

$$\Gamma^r_{\theta\theta} = -r,\ \Gamma^r_{\phi\phi} = -r\sin^2\theta,\ \Gamma^\theta_{r\theta} = \frac{1}{r},\ \Gamma^\phi_{r\phi} = \frac{1}{r},\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\ \text{and}\ \Gamma^\phi_{\theta\phi} = \cot\theta,$$

all other being zero, where we have used: $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$, and the symmetry: $\Gamma^i_{jk} = \Gamma^i_{(jk)}$.

c) Find as many independent constants of the motion as you can (and compare with the situation in quantum mechanics).

It is obvious that $L_z = mr^2\sin^2\theta\,\dot{\phi}$ is conserved. Not so obvious is the fact that L_x and L_y are also conserved. Even less obvious is the fact that p_x, p_y and p_z are all conserved too, as is E_{Total}.

3a) Show for a differentiable function φ defined over a manifold that the set $\{\partial\varphi/\partial x^a\}$ transforms as do the components of a tensor of rank one; what type of tensor has this construction defined?

We find:

$$\frac{\partial\varphi}{\partial x^{a'}} = \frac{\partial x^a}{\partial x^{a'}}\frac{\partial\varphi}{\partial x^a},$$

which is the transformation law for a covector.

b) Write out the components of this tensor in polar coordinates explicitly in terms of its components in cartesian coordinates.

We calculate, $\nabla\varphi = \left(\varphi_{,r}, \varphi_{,\theta}, \varphi_{,\phi}\right)$, where:

$$\begin{aligned}
\varphi_{,r} &= x_{,r}\varphi_{,x} + y_{,r}\varphi_{,y} + z_{,r}\varphi_{,z} \\
&= \sin\theta\cos\phi\,\varphi_{,x} + \sin\theta\sin\phi\,\varphi_{,y} + \cos\theta\,\varphi_{,z} \\
\varphi_{,\theta} &= x_{,\theta}\varphi_{,x} + y_{,\theta}\varphi_{,y} + z_{,\theta}\varphi_{,z} \\
&= r\cos\theta\cos\phi\,\varphi_{,x} + r\cos\theta\sin\phi\,\varphi_{,y} - r\sin\theta\,\varphi_{,z} \\
\varphi_{,\phi} &= x_{,\phi}\varphi_{,x} + y_{,\phi}\varphi_{,y} + z_{,\phi}\varphi_{,z} \\
&= -r\sin\theta\sin\phi\,\varphi_{,x} + r\sin\theta\cos\phi\,\varphi_{,y}
\end{aligned}$$

c) Compare this with the expression you have previously used for the gradient of a function in spherical polar coordinates, and comment on any differences you see.

The usual result quoted in Electromagnetic contexts (*e.g.,* in Griffiths, Jackson, *etc.*) would be:

$$\nabla\varphi = \left(\varphi_{,r}, \frac{1}{r}\varphi_{,\theta}, \frac{1}{r\sin\theta}\varphi_{,\phi}\right),$$

for which the magnitude squared would be calculated using an orthonormal metric, not the metric in spherical polar coordinates.

4) Maxwell's equations for electromagnetism may be written in a geometrically covariant form as:

$$\nabla_i B^i = 0,\ \nabla_i E^i = 4\pi\rho,\ \varepsilon^{ijk}\nabla_j E_k = -\dot{B}^i,\ \varepsilon^{ijk}\nabla_j B_k = \dot{E}^i + 4\pi J^i,$$

where ∇_i is a derivative operator (generally not $\partial/\partial x^i$), and E^i is not to be confused with E_i. In accord with these equations, the *electromagnetic stress tensor* T_i^j may be defined according to the formula:

$$4\pi T_i^j = -E_i E^j - B_i B^j + \frac{1}{2}\delta_i^j (E_k E^k + B_k B^k).$$

a) Try to show that, in a vacuum, it follows as a consequence of Maxwell's equations that

$$\dot{P}_i + \nabla_j T_i^j = 0,$$

where P_i is the electromagnetic energy flux vector defined by

$$4\pi P_i = \varepsilon_{ijk} E^j B^k.$$

Indicate as precisely as you can what extra information you would need to complete the proof.

Substituting, we have:

$$\begin{aligned}
4\pi \nabla_j T_i^j &= -\left(\nabla_j E_i\right) E^j - E_i \left(\nabla_j E^j\right) - \left(\nabla_j B_i\right) B^j - B_i \left(\nabla_j B^j\right) + \\
&\quad \frac{1}{2}\left[\left(\nabla_i E_k\right) E^k + E_k \left(\nabla_i E^k\right) + \left(\nabla_i B_k\right) B^k + B_k \left(\nabla_i B^k\right)\right], \quad \text{and} \\
4\pi \dot{P}_i &= \varepsilon_{ijk} \dot{E}^j B^k + \varepsilon_{ijk} E^j \dot{B}^k \\
&= \varepsilon_{ijk} \left(\varepsilon^{jlm} \nabla_l B_m\right) B^k - \varepsilon_{ijk} E^j \left(\varepsilon^{klm} \nabla_l E_m\right) \\
&= \left(\delta_k^l \delta_i^m - \delta_i^l \delta_k^m\right) \left(\nabla_l B_m\right) B^k - \left(\delta_i^l \delta_j^m - \delta_j^l \delta_i^m\right) E^j \left(\nabla_l E_m\right) \\
&= \left(\nabla_k B_i\right) B^k - \left(\nabla_i B_k\right) B^k - E^j \left(\nabla_i E_j\right) + E^j \left(\nabla_j E_i\right).
\end{aligned}$$

Adding, we find:

$$4\pi \left(\dot{P}_i + \nabla_j T_i^j\right) = -\frac{1}{2}\left[\left(\nabla_i E_k\right) E^k - E_k \left(\nabla_i E^k\right) + \left(\nabla_i B_k\right) B^k - B_k \left(\nabla_i B^k\right)\right].$$

To complete the proof, the first two terms and the last two terms would have to separately cancel.

b) Suppose there exists a symmetric second rank tensor g_{ij} which allows you to relate the covector E_i to the contravariant vector E^i via $E_i = g_{ij} E^j$. If this is to be enough extra information to prove the above conservation equations, what can you deduce about $\nabla_k g_{ij}$?

Substituting, we compute $\nabla_i E_k = \left(\nabla_i g_{kj}\right) E^j + g_{kj}\left(\nabla_i E^j\right)$, and $\nabla_i B_k = \left(\nabla_i g_{kj}\right) B^j + g_{kj}\left(\nabla_i B^j\right)$, and hence:

$$4\pi \left(\dot{P}_i + \nabla_j T_i^j\right) = -\frac{1}{2}\left(\nabla_i g_{kj}\right)\left(E^j E^k + B^j B^k\right),$$

where we have used the symmetry of g_{kj} to replace $g_{kj} E^k$ byE_j, *etc.* before canceling.

Questions for PHZ 6607, Special and General Relativity I

5a) Using the Lagrangian

$$\pounds = \frac{1}{2} m g_{\mu\nu} \dot{x^\mu} \dot{x^\nu},$$

in cartesian (spatial) coordinates, find expressions for the conserved linear and angular canonical momenta p_x, p_y, p_z, L_x, L_y and L_z, in terms of the 'velocities'.

We have $\pounds = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, and find:

$$\begin{aligned}
p_x &= \frac{\partial \pounds}{\partial \dot{x}} = m\,\dot{x}, \\
p_y &= \frac{\partial \pounds}{\partial \dot{y}} = m\,\dot{y}, \\
p_z &= \frac{\partial \pounds}{\partial \dot{z}} = m\,\dot{z}, \quad \text{and, by definition,} \\
L_x &= y p_z - z p_y = m(y\dot{z} - z\dot{y}), \\
L_y &= z p_x - x p_z = m(z\dot{x} - x\dot{z}), \\
L_z &= x p_y - y p_x = m(x\dot{y} - y\dot{x}).
\end{aligned}$$

b) Similarly, starting with the Lagrangian written in spherical polar coordinates, find expressions for the canonical momenta p_r, p_θ and p_ϕ in terms of the 'velocities' which arise for these coordinates.

We have $\pounds = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2)$, and find:

$$\begin{aligned}
p_r &= \frac{\partial \pounds}{\partial \dot{r}} = m\,\dot{r}, \\
p_\theta &= \frac{\partial \pounds}{\partial \dot{\theta}} = m r^2\,\dot{\theta}, \quad \text{and} \\
p_\phi &= \frac{\partial \pounds}{\partial \dot{\phi}} = m r^2 \sin^2\theta\,\dot{\phi}.
\end{aligned}$$

c) Use the coordinate relations:

$$\begin{aligned}
x &= r \sin\theta \cos\phi, \\
y &= r \sin\theta \sin\phi, \;\text{ and} \\
z &= r \cos\theta,
\end{aligned}$$

write down the relations giving the cartesian velocities in terms of the polar velocities.

We calculate:

$$\begin{aligned}
\dot{x} &= \sin\theta\cos\phi\,\dot{r} + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}, \\
\dot{y} &= \sin\theta\sin\phi\,\dot{r} + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}, \\
\dot{z} &= \cos\theta\,\dot{r} - r\sin\theta\,\dot{\theta}.
\end{aligned}$$

d) Hence, obtain expressions for all the conserved momenta listed in part a) in terms of the canonical momenta for polar coordinates.

For the linear momenta, we find simply:

$$\begin{aligned}
p_x &= m(\sin\theta\cos\phi\,\dot r + r\cos\theta\cos\phi\,\dot\theta - r\sin\theta\sin\phi\,\dot\phi)\\
&= \sin\theta\cos\phi\,p_r + \frac{\cos\theta\cos\phi}{r}\,p_\theta - \frac{\sin\phi}{r\sin\theta}\,p_\phi,\\
p_y &= m(\sin\theta\sin\phi\,\dot r + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi)\\
&= \sin\theta\sin\phi\,p_r + \frac{\cos\theta\sin\phi}{r}\,p_\theta + \frac{\cos\phi}{r\sin\theta}\,p_\phi,\\
p_z &= m(\cos\theta\,\dot r - r\sin\theta\,\dot\theta)\\
&= \cos\theta\,p_r - \frac{\sin\theta}{r}\,P_\theta.
\end{aligned}$$

For the angular momenta, we find:

$$\begin{aligned}
L_x &= mr\sin\theta\sin\phi(\cos\theta\,\dot r - r\sin\theta\,\dot\theta) - mr\cos\theta(\sin\theta\sin\phi\,\dot r + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi)\\
&= -mr^2(\sin\phi\,\dot\theta + \sin\theta\cos\theta\cos\phi\,\dot\phi) = -\sin\phi\,p_\theta - \cot\theta\cos\phi\,p_\phi,\\
L_y &= mr\cos\theta(\sin\theta\cos\phi\,\dot r + r\cos\theta\cos\phi\,\dot\theta - r\sin\theta\sin\phi\,\dot\phi) - mr\sin\theta\cos\phi(\cos\theta\,\dot r - r\sin\theta\,\dot\theta),\\
&= mr^2(\cos\phi\,\dot\theta - \sin\theta\cos\theta\sin\phi\,\dot\phi) = \cos\phi\,p_\theta - \cot\theta\sin\phi\,p_\phi,\\
L_z &= mr\sin\theta\cos\phi(\sin\theta\sin\phi\,\dot r + r\cos\theta\sin\phi\,\dot\theta + r\sin\theta\cos\phi\,\dot\phi)\\
&\quad - mr\sin\theta\sin\phi(\sin\theta\cos\phi\,\dot r + r\cos\theta\cos\phi\,\dot\theta - r\sin\theta\sin\phi\,\dot\phi)\\
&= mr^2\sin^2\theta\,\dot\phi = p_\phi,
\end{aligned}$$

the last of which is exactly p_ϕ as computed in b) above.

e) What type of tensor are the velocities and what type the momenta? (This difference explains why it is dangerous to go directly from the velocities in one set of coordinates to the momenta in another. It also explains why $\mathbf{p}\cdot\mathbf{x}$ is always a scalar.)

The velocities are tensors of rank $(1,0)$ and the momenta are tensors of rank $(0,1)$.

6) Consider the four-dimensional spacetime metric which in $\{t, r, \theta, \phi\}$ coordinates has diagonal components $\{-(1-r_s/r), 1/(1-r_s/r), r^2, r^2\sin^2\theta\}$, all other components being zero. Notice that in the usual sense, the parameter r_s has the dimensions of length.

a) Write out expressions for the canonical momenta of a particle whose Lagrangian is given by:

$$\pounds = \frac{1}{2}mg_{\mu\nu}\dot{x^\mu}\dot{x^\nu},\ \text{in which}\ \dot{} = d/d\lambda,$$

in which the spacetime geometry is given by this metric.

Calculating (and restoring c^2), we find:

$$\begin{aligned}
p_t &= -mc^2\left(1-\frac{r_s}{r}\right)\dot t,\\
p_r &= \frac{m\,\dot r}{\left(1-\frac{r_s}{r}\right)},\\
p_\theta &= mr^2\,\dot\theta,\\
p_\phi &= mr^2\sin^2\theta\,\dot\phi.
\end{aligned}$$

b) Write out Hamilton's equations of motion for this particle, and indicate which canonical momenta are conserved.

Solving for the velocities and substituting, we obtain:

$$\begin{aligned}\mathcal{H} &= p_\mu \dot{x}^\mu = \mathcal{L} = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu \\ &= -\frac{1}{2mc^2} \frac{p_t^2}{\left(1-\frac{r_s}{r}\right)} + \frac{1}{2m}\left(1-\frac{r_s}{r}\right) p_r^2 + \frac{1}{2mr^2} p_\theta^2 + \frac{1}{2mr^2 \sin^2\theta} p_\phi^2.\end{aligned}$$

Hamilton's equations give:

$$\begin{aligned}\dot{p}_\mu &= -\frac{\partial \mathcal{H}}{\partial x^\mu}, \\ \dot{x}^\mu &= +\frac{\partial \mathcal{H}}{\partial p_\mu}.\end{aligned}$$

Since the metric is independent of both t and ϕ, both $p_t = -E$ and $p_\phi = L_z$ are conserved. Thus, the non-trivial equations become:

$$\begin{aligned}\dot{p}_r &= -\frac{r_s}{2mr^2c^2} \frac{p_t^2}{\left(1-\frac{r_s}{r}\right)^2} - \frac{r_s}{2mr^2} p_r^2 + \frac{1}{mr^3} p_\theta^2 + \frac{1}{mr^3 \sin^2\theta} p_\phi^2, \\ \dot{p}_\theta &= \frac{\cos\theta}{mr^2 \sin^3\theta} p_\phi^2.\end{aligned}$$

c) By explicit calculation, show that all components of the angular momentum which were defined in Question 5) are in fact conserved for the particle moving in this non-flat geometry. Hence, or otherwise, argue whether or not this spacetime is spherically symmetric.

From above we find:

$$\begin{aligned}L_x &= -\sin\phi p_\theta - \cot\theta \cos\phi p_\phi, \\ L_y &= +\cos\phi p_\theta - \cot\theta \sin\phi p_\phi, \\ L_z &= p_\phi.\end{aligned}$$

Note: $L^2 = p_\theta^2 + p_\phi^2/\sin^2\theta$. Computing, we find:

$$\begin{aligned}\dot{L}_x &= -\cos\phi \frac{p_\phi p_\theta}{mr^2 \sin^2\theta} - \sin\phi \frac{\cos\theta p_\phi^2}{mr^2 \sin^3\theta} + \frac{\cos\phi}{\sin^2\theta} \frac{p_\theta p_\phi}{mr^2} + \cot\theta \sin\phi \frac{p_\phi^2}{mr^2 \sin^2\theta} = 0, \\ \dot{L}_y &= -\sin\phi \frac{p_\phi p_\theta}{mr^2 \sin^2\theta} + \cos\phi \frac{\cos\theta p_\phi^2}{mr^2 \sin^3\theta} + \frac{\sin\phi}{\sin^2\theta} \frac{p_\theta p_\phi}{mr^2} - \cot\theta \cos\phi \frac{p_\phi^2}{mr^2 \sin^2\theta} = 0.\end{aligned}$$

Spherical symmetry follows from the form of the metric; it also follows from the conservation of all three angular momenta.

d) Find an expression for $dr/d\phi$, assuming the motion takes place in the equatorial plane. Obtain the corresponding equation for a particle of mass m moving under Newtonian gravity around a central body of mass M.

In the equatotial plane, $\sin\theta = 1$ and $L = L_z$. To find an expression for $dr/d\phi$ it is simplest to start from the "conservation of momentum" $p_\mu p_\nu g^{\mu\nu} = -m^2c^2$, which gives:

$$-\frac{E^2}{c^2\left(1-\frac{r_s}{r}\right)} + \frac{m^2 \dot{r}^2}{\left(1-\frac{r_s}{r}\right)} + \frac{L^2}{r^2} = -m^2c^2.$$

We can rearrange this to find $\dot{r}$, via:

$$\dot{r}^2 = \frac{E^2}{m^2c^2} - \left(1 - \frac{r_s}{r}\right)\left(c^2 + \frac{L^2}{m^2r^2}\right).$$

Using $\dot{\phi} = L/mr^2$ (recall $\sin\theta = 1$ and $L_z = L$), expanding and dividing, we finally obtain:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{m^2r^4}{L^2}\left[\frac{E^2}{m^2c^2} - c^2 + \frac{r_sc^2}{r} - \frac{L^2}{m^2r^2} + \frac{r_sL^2}{m^2r^3}\right].$$

In the Newtonian case we would have correspondingly:

$$E_{\rm TOT} = \frac{1}{2}\left(m\dot{r}^2 + \frac{L^2}{mr^2}\right) - \frac{GMm}{r}, \quad \text{and} \quad \dot{\phi} = \frac{L}{mr^2},$$

where $\dot{}$ now means d/dt rather than $d/d\tau$. Solving for $\dot{r}$ and dividing, we obtain:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{m^2r^4}{L^2}\left[\frac{2E_{\rm TOT}}{m} + \frac{2GM}{r} - \frac{L^2}{m^2r^2}\right].$$

e) Using your knowledge of the difference between Newtonian physics and relativistic physics, and by assuming that the motion takes place in a region where both $|\mathbf{v}|/c$ the ratio r_s/r are small, indicate how the resulting motion in this spacetime can be compared with motion in a Newtonian gravitational potential, and find the implied correspondence between the parameter r_s and the mass causing the potential: if you have used $c = 1$, restore it in the final relation you obtain.

In the relativistic case, let us replace E by $mc^2 + \Delta E$ and assume $\Delta E/mc^2 \ll 1$. Then, we have:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{m^2r^4}{L^2}\left[\frac{2\Delta E}{m} + \frac{\Delta E^2}{m^2c^2} + \frac{r_sc^2}{r} - \frac{L^2}{m^2r^2} + \frac{r_sL^2}{m^2r^3}\right].$$

Now, the first term here corresponds almost directly to the first term in the Newtonian case, and the second term here represents a small relativistic correction to it. The third term here can be compared with the second term in the Newtonian case since they both have a $1/r$ dependance. These terms are exactly equal if $r_s = 2GM/c^2$. The fourth term here corresponds exactly to the third term in the Newtonian case, and the fifth term here is a small relativistic correction to the fourth term, since now $r_s/r \sim 2(|\mathbf{v}|/c)^2$, and both ratios are assumed small.