PHY 6607 Special and General Relativity

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1. Diffeomorphisms

This lecture is a continuation of the previous lecture in which we discussed maps between manifolds and submanifolds. We follow Appendix B from S. Carroll.

Recall the general case from last class: We had a manifold M which mapped to the reals R^m, and a manifold N which mapped to R^n, and a map ϕ from M to N (Fig. 1).

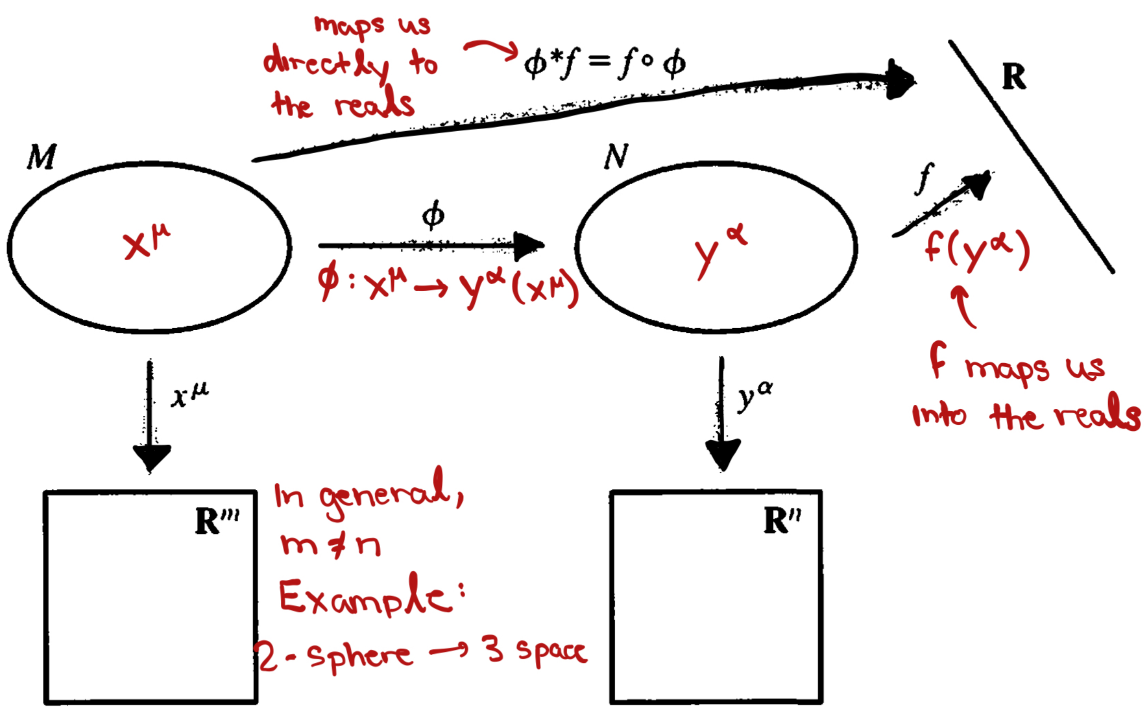

Figure 1.

Today, we will look at the special case when $n = m$ and $N = M$ (Fig. 2). That is, the two manifolds are actually the same. Recall from last class, we defined the **pullback** of f by ϕ by:

$$\phi^* f = (f \circ \phi).$$

We also gave a matrix description of the pullback operator:

$$(\phi^*)^\alpha_\mu = dy^\alpha / dx^\mu.$$

If ϕ is a map from M to N, $\phi : M \to N$, and ϕ is invertible, then it defines a **diffeomorphism** between M and N.

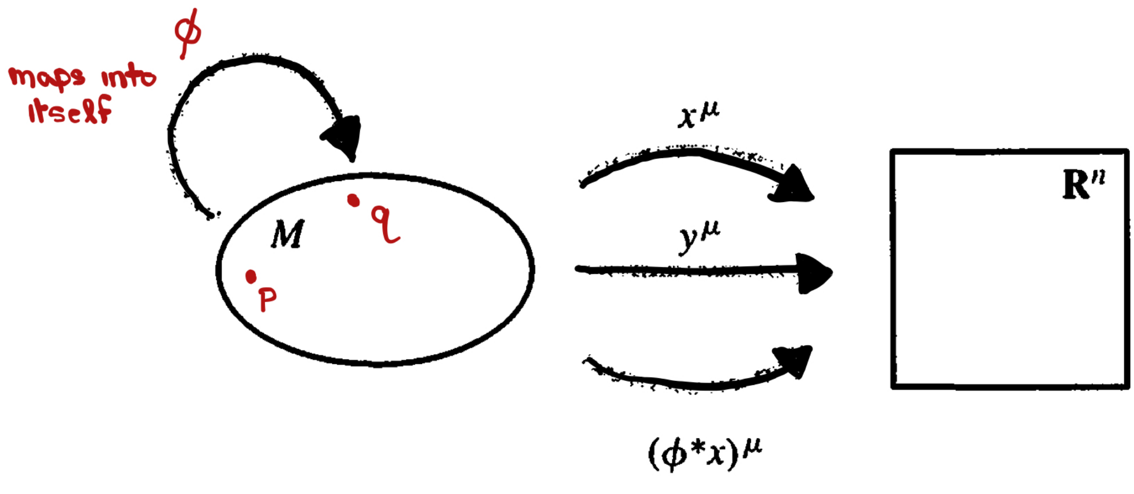

Figure 2.

We will evaluate this at some point $p \epsilon M$. Since, ϕ maps $M \to M$, it follows that ϕ maps $p \to q \epsilon M$. When we look at pullbacks, we are able to associate tensors at q to tensors evaluated at p.

1.1. Example: 2D Caertesian

Let's begin to understand this by considering a trivial yet concrete example. Consider a simple translation along the y-axis in 2D Caertesian coordinates:

$$x' = x,$$

$$y' = y + 2.$$

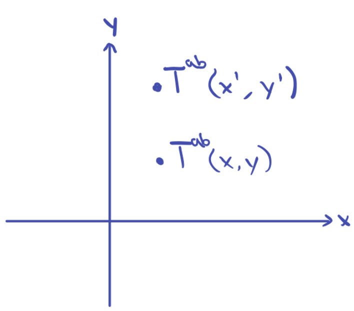

Figure 3.

If we have some tensor T^{ab}, the transformation of the tensor looks like:

$$(T^{ab})'(x, y) = T^{ab}(x' = x, y' = y + 2).$$

Something interesting to consider is:

$$\Delta T^{ab} = (T^{ab})' - T^{ab}.$$

We can associate this change with a vector ξ, where for our example, we have $\xi = (0, 2)$. So we've gone from coordinates x^μ to $x^{\mu'} = x^\mu + \xi$. This is similar to what we did for parallel transport, and it's useful to think of a one parameter family of these.

In evaluating $(T^{ab})'$, there are two steps: (i) change from the coordinates x^μ to $x^{\mu'}$, and (ii) also move the point in the tensor. The change discussed above leads us to the definition of the **Lie derivative** of a tensor T^{ab} in the direction of some vector ξ:

$$\frac{\Delta T^{ab}}{t} = \frac{(T^{ab})' - T^{ab}}{t} = \mathcal{L}_\xi T.$$

Diffeomorphisms are closely related to coordinate transformations. They involve a coordinate transformation from p to q, but they also involve a map from the point p to the point q of tensors at p to tensors at q.

We looked at examples of the derivatives already. But let's look at how they act on different objects.

1.2. Lie derivative acting on vectors and tensors

Suppose we have the tensor V^μ. We want to evaluate the Lie derivative this tensor $\mathcal{L}_\xi V^\mu$. Let's suppose we take $\xi = (0, \xi^y)$. Some vector with component only in the y-direciton. So we have:

$$x' = x,$$

$$y' = y + t\,\xi^y.$$

Let's consider the partial derivative:

$$\frac{\partial V^\mu}{\partial y} = lim_{t\to 0}\frac{V^\mu(x, y + t\,\xi^y) - V^\mu(x, y)}{t}.$$

So the Lie derivative of vector V^μ is:

$$\mathcal{L}_\xi V^\mu = \xi^\nu \nabla_\nu V^\mu - V^\nu \nabla_\nu \xi^\mu,$$

$$\mathcal{L}_\xi V^\mu = \xi^\nu \partial_\nu V^\mu - V^\nu \partial_\nu \xi^\mu,$$

$$\mathcal{L}_\xi V^\mu = \xi^\nu \partial_\nu V^\mu.$$

The action of the Lie derivative on a one-form is:

$$\mathcal{L}_\xi \omega_\mu = \xi^\nu \nabla_n u \omega_\mu + \omega_\nu \nabla_\mu \xi^\nu.$$

The action of the Lie derivative on the metric is:

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\sigma \nabla_\sigma g_{\mu\nu} + g_{\sigma\nu} \nabla_\mu \xi^\sigma + g_{\mu\sigma} \nabla_\nu \xi^\sigma$$

$$\mathcal{L}_\xi g_{\mu\nu} = g_{\sigma\nu} \nabla_\mu \xi^\sigma + g_{\mu\sigma} \nabla_\nu \xi^\sigma$$

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu g_{\sigma\nu} \xi^\sigma + g_{\mu\sigma} \nabla_\nu \xi^\sigma$$

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu - \xi^\sigma \nabla_\mu g_{\mu\nu} + \nabla_\nu \xi_\mu - \xi^\sigma \nabla_\nu g_{\mu\sigma}$$

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.$$

Note that we used properties of the metric and this is not true for an ordinary tensor. What is this telling us? It says that the metric transforms as $g_{\mu\nu} \to g_{\mu\nu} + t(\nabla_\mu \xi_\nu - \nabla_\nu \xi_\nu)$.

So there is a one-to-one correspondance between the metric (in general all tensors) at p to the metric (all tensors) at q i.e., there is a map from tensor at q to same tensor at p. We notice that this arrangement is of the "Killing form" as it matches the form we associated with Killing vectors. So suppose ξ^μ is a Killing vector K^μ. We find that $\Delta g = \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0$. So the metric is left unchanged by the action of K^μ. We will find that the scalar curvature R also does not change in the homework.

Before we return to Killing vectors, more on the stress tensor.

1.3. The stress tensor

If we have some action: $S = \frac{1}{16\pi G} S_{EH}$ (Einstein Hilbert action) $+S_\mu$. We consider the total variation:

$$\delta S = \frac{1}{16\pi G}\Big(\frac{\delta S_{EH}}{\delta g^{\mu\nu}} + 16\pi G \frac{\delta S_\mu}{\delta g^{\mu\nu}}\Big)\delta g^{\mu\nu} + \frac{\delta S_\mu}{\delta \phi_\mu}\delta\phi_\mu.$$

With some normalization, the part $\frac{\delta S_{EH}}{\delta g^{\mu\nu}}$ becomes the Einstein tensor $G^{\mu\nu}$ and the part $\frac{\delta S_\mu}{\delta g^{\mu\nu}}$ becomes the stress tensor $T^{\mu\nu}$. Essentially, what we have in parentheses in the above equation is: $G^{\mu\nu} - 8\pi G T^{\mu\nu} = 0$ (the Einstein equation). The argument for why this is zero is that we want it to be extremal under any metric perturbation, and so the coefficient must be zero.

Now if we consider:

$$(G^{\mu\nu} - 8\pi G T^{\mu\nu})(\Delta g_{\mu\nu})$$

$$= (G^{\mu\nu} - 8\pi G T^{\mu\nu})(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)$$

integrating by parts:

$$= -2(\nabla_\mu G^{\mu\nu} - 8\pi G \nabla_\mu T^{\mu\nu})\xi_\nu.$$

For arbitrary ξ_ν, $\nabla_\mu G^{\mu\nu} - 8\pi G \nabla_\mu T^{\mu\nu} = 0$. We know independently that $\nabla_\mu G^{\mu\nu} = 0$. So this implies: $\nabla_\mu T^{\mu\nu} = 0$. So, the invariance of the metric under the action of abritrary vectors implies that the stress tensor for matter is conserved.