Special and General Relativity Notes

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1 Directional Derivatives

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu} \tag{1}$$

1.1 Direction Covariant Derivative

$$\begin{aligned}\frac{DV^\mu}{d\lambda} &= \frac{dx^\mu}{d\lambda}\nabla_\mu V^\mu \\ &= \frac{dx^\mu}{d\lambda}\left(\frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\sigma}V^\sigma\right) \\ &= \frac{dV^\mu}{d\lambda} + \Gamma^\nu_{\mu\sigma}V^\sigma\frac{dx^\mu}{d\lambda}\end{aligned} \tag{2}$$

where

$$\frac{dx^\mu}{d\lambda}\frac{\partial V^\nu}{\partial x^\mu} = \frac{dV^\mu}{d\lambda}$$

For some vector U:

$$\frac{DV^\mu}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu V^\mu \equiv U^\mu \nabla_\mu V^\mu$$

Meaning that:

$$\frac{dx^\mu}{d\lambda} = U^\mu \tag{3}$$

1.1.1 Parallel Transport

For all U^μ in which Eq. 3 is true there exists a $V^\nu(\lambda)$ such that

$$U^\mu \nabla_\mu V^\nu(\lambda) = 0.$$

This is a parallel transport in which the path is given and we solve to find $V^\nu(\lambda)$. Parallel transport allows you to compare a vector in one tangent plane to a vector in another. This is done by moving the vector along a curve without changing it.

1.1.2 Geodesics

A geodesic is a curve along which U^{μ} is preserved such that

$$U^{\mu}\nabla_{\mu}U^{\nu} = 0.$$

This is a second order differential equation and you would solve for the path. A geodesic is a length minimizing curve. In a plane this would be a straight line and on a sphere this is a great circle.

$$\begin{aligned} \frac{D\left(U^{\mu}\right)}{d\lambda} &= \frac{dx^{\mu}}{d\lambda}\frac{\partial}{\partial x^{\mu}}\left(\frac{dx^{\nu}}{d\lambda}\right) + \Gamma^{\nu}_{\mu\sigma}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} \\ &= \frac{d^2x^{\nu}}{d\lambda^2} + \Gamma^{\nu}_{\mu\sigma}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\sigma}}{d\lambda} = 0 \end{aligned} \tag{4}$$

The equation $\lambda \to \lambda' = a\lambda + b$ stays the same and rescales the velocity vector:

$$U^{\mu} \to U'^{\mu} = \frac{1}{a}U^{\mu}$$

Any vector at a point is the initial velocity of a geodesic with some parameterization. At any point p there exists a velocity U^{ν} in the tangent space TpM that is $U^{\nu} = dx^{\nu}/d\lambda$ for some λ along a geodesic.

1.2 Applying Directional Derivatives

For the case of the sphere the metric is given by:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

In which the connection coefficients are:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad\qquad \Gamma^{\phi}_{\phi\theta} = \Gamma^{\phi}_{\theta\phi} = \frac{\cos\theta}{\sin\theta}$$

For the sphere in Fig. 1:

$$V^{\nu}(\lambda) = \left(V^{\theta}(\lambda), V^{\phi}(\lambda)\right)$$

At $\lambda = 0$ this gives:

$$\begin{aligned} V^{\nu}(0) &= \left(V^{\theta}(0), V^{\phi}(0)\right) \\ &= \left(V^{\theta}_0, V^{\phi}_0\right) \end{aligned}$$

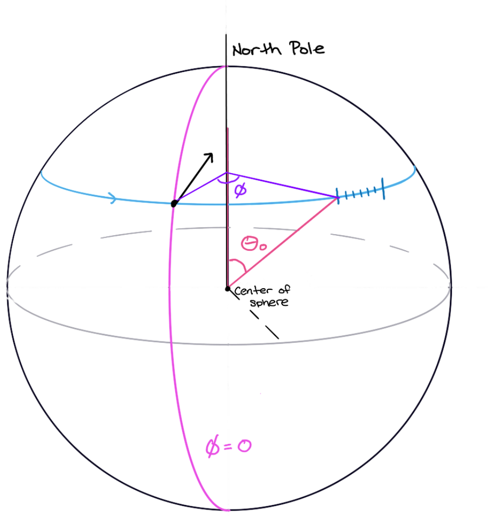

Figure 1: Looking at this sphere, the blue line is the path we have decided to take where $\theta_0 \neq 0$. The black line is the vector in question that we will be watching to see how its direction changes as it moves along the path.

λ is some parameter along a decided monotonic path. In this case we choose $\lambda = \phi$. Since R is fixed we get two equations using Eq. 2 with $dx^\mu/d\lambda = 1$ since we chose $x^\mu = \phi = \lambda$:

$$\frac{DV^\theta}{d\phi} = \frac{dV^\theta}{d\phi} + \Gamma^\theta_{\phi\phi} V^\phi = 0 \qquad \frac{DV^\phi}{d\phi} = \frac{dV^\phi}{d\phi} + \Gamma^\phi_{\theta\phi} V^\theta = 0 \tag{5}$$

These are examples of parallel transport. We will solve these equations to determine how V changes in ϕ increments.

Plugging in the connection coefficients for $\theta = \theta_0$:

$$\frac{dV^\theta}{d\lambda} - \sin\theta_0 \cos\theta_0 V^\phi = 0 \qquad \frac{dV^\phi}{d\lambda} + \frac{\cos\theta_0}{\sin\theta_0} V^\theta = 0 \tag{6}$$

Now taking the derivative:

$$\frac{d^2V^\theta}{d\lambda^2} - \sin\theta_0 \cos\theta_0 \frac{dV^\phi}{d\lambda} = 0$$
$$\frac{d^2V^\theta}{d\lambda^2} - \sin\theta_0 \cos\theta_0 \frac{\cos\theta_0}{\sin\theta_0} V^\theta = 0$$
$$\frac{d^2V^\theta}{d\lambda^2} - \cos^2(\theta_0) V^\theta = 0 \tag{7}$$

Solving Eq. 7 we get equations for V^θ and V^ϕ:

$$V^\theta = A\cos\left[\phi\cos(\theta_0)\right] + \frac{B}{\cos(\theta_0)} \sin\left[\phi\cos(\theta_0)\right] \tag{8}$$

$$V^\phi = C\cos\left[\phi\cos(\theta_0)\right] + \frac{D}{\cos(\theta_0)} \sin\left[\phi\cos(\theta_0)\right] \tag{9}$$

Let $\phi = 0$ then solving Eqs. 8 and 9 gives $A = V_0^\theta$ and $C = V_0^\phi$. Taking the derivative of these equations with respect to λ:

$$\frac{dV^\theta}{d\lambda} = \cos\theta_0 A \sin(\phi\cos\theta_0) + B\cos(\phi\cos\theta_0)$$
$$\frac{dV^\phi}{d\lambda} = \cos\theta_0 C \sin(\phi\cos\theta_0) + D\cos(\phi\cos\theta_0)$$

Setting $\phi = \lambda = 0$ we can then use Eqs. 6 at to determine the other constants.

$$B = \left.\frac{dV^\theta}{d\lambda}\right|_{\phi=0} \qquad\qquad D = \left.\frac{dV^\phi}{d\lambda}\right|_{\phi=0}$$
$$= \sin\theta_0\cos\theta_0 V_0^\phi \qquad\qquad = -\frac{\cos\theta_0}{\sin\theta_0}V_0^\theta$$

The final result then gives an example of a geodesic:

$$V^\theta(\phi) = V_0^\theta\cos(\phi\cos\theta_0) + \sin\theta_0 V_0^\phi\sin(\phi\cos\theta_0) \tag{10}$$
$$V^\phi(\phi) = V_0^\phi\cos(\phi\cos\theta_0) - \frac{1}{\sin\theta_0}V_0^\theta\sin(\phi\cos\theta_0) \tag{11}$$

Now looking at two cases, one where $\theta_0 = \pi/2$ and the other where $\theta_0 = \pi/3$.

Case 1:
If $\theta_0 = \pi/2$ then $\sin\theta_0 = 1$ and $\cos\theta_0 = 0$ giving

$$V^\theta(\phi) = V_0^\theta \qquad\qquad V^\phi(\phi) = V_0^\phi$$

for all ϕ

Case 2:
If $\theta_0 = \pi/3$ then $\sin\theta_0 = \sqrt{3}/2$ and $\cos\theta_0 = 1/2$ giving

$$V^\theta(\phi) = V_0^\theta\cos\left(\frac{\phi}{2}\right) + \frac{\sqrt{3}}{2}V_0^\phi\sin\left(\frac{\phi}{2}\right)$$
$$V^\phi(\phi) = V_0^\phi\cos\left(\frac{\phi}{2}\right) + \frac{2}{\sqrt{3}}V_0^\theta\sin\left(\frac{\phi}{2}\right)$$

For $\phi = 0$ we would get the same result as in Case 1, but if $\phi = 2\pi$ then we would get negative of that result. The vector will be in the opposite direction once it goes around the curve once.

2 Exponential Map

Given a point p and some vector V_p^ν in the tangent space, there exists a geodesic $\gamma_p(\lambda)$ such that $\gamma_p(0) = p$ and $\dot{\gamma}_p(0) = V_p^\nu$.

Construct a 1D space where:

$$E = (p, V_p^\nu, \gamma_p(\lambda) = \gamma_p(\lambda, p, V_p^\nu), \lambda = (0, 1))$$

This includes all points and lines along a geodesic. The different points come from different values for t, $V'^\nu = (1/t)V^\nu$.
Note: If $\lambda \to \lambda' = t\lambda$ where t exists between 0 and 1, then for different t the endpoints for λ' change.

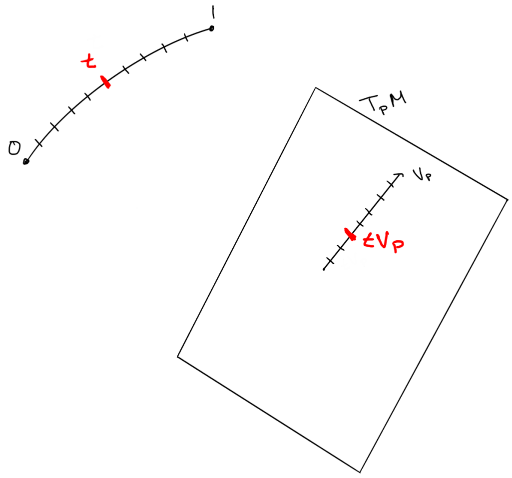

Figure 2: This shows the map from the flat tangent space of the manifold (TpM) to the manifold, defining an "exponential" map.

Looking at Fig. 2 we now have a map from the tangent space of the manifold to the manifold.

What defines an "exponential" map:

$$\begin{aligned} \exp_p(V_p^\nu) &= \gamma(1, p, V_p^\nu) \\ \exp_p(tV_p^\nu) &= \gamma(1, p, tV_p^\nu) \end{aligned} \tag{12}$$

Eq. 12 scales to every point along the curve. There may be restrictions on how much of the manifold can be mapped using this method.